Explicit solutions of the WDVV equation determined by the "flat" hydrodynamic reductions of the Egorov hydrodynamic chains.

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Abstract

Classification of the Egorov hydrodynamic chain and corresponding 2+1 quasilinear system is given in [33]. In this paper we present a general construction of explicit solutions for the WDVV equation associated with Hamiltonian hydrodynamic reductions of these Egorov hydrodynamic chains.

Contents

keywords: hydrodynamic chain, Hamiltonian structure, Miura type transformation, hydrodynamic reduction, hydrodynamic type system, WDVV equation.

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1 Introduction

The associativity equations (or WDDV equations) appeared in the classification problem for the topological field theories at the early 90's, see [8] and [9]. During the last years these equations has attracted a great interest due to connections with the enumerative geometry (Gromov–Witten invariants [23]), quantum cohomology [40], the Whitham theory [24] and centroaffine geometry [14].

This paper is devoted to a construction of **explicit** solutions (expressed via elementary or well-known special functions of flat coordinates) of the WDVV equation (see [9]) connected with hydrodynamic reductions of the Egorov hydrodynamic chains (see [33]). For simplicity we restrict our consideration on the class of N component Hamiltonian hydrodynamic type systems embedded in the famous Benney hydrodynamic chain (see [1], [3], [20], [24], [26])

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \qquad k = 0, 1, 2, ... \tag{1}$$

and in the Kupershmidt hydrodynamic chains (see [25], [32])

$$B_t^k = B_x^{k+1} + B^0 B_x^k + (\beta k + \gamma) B^k B_x^0. \tag{2}$$

Definition: *The semi-Hamiltonian* (see [42]) *hydrodynamic type system*

$$r_t^i = v^i(\mathbf{r}) r_x^i, \qquad i = 1, 2, ..., N \tag{3}$$

possessing the couple of conservation laws

$$a_t = b_x, \qquad b_t = c_x \tag{4}$$

is said to be ***Egorov*** *(see* [38]*; and also* [34]).

It means, that the conservation law density a is a potential of the Egorov diagonal metric $g_{ii} = H_i^2$, where the Lame coefficients H_i can be found from (see [42])

$$\partial_i \ln H_k = \frac{\partial_i v^k}{v^i - v^k}, \qquad i \neq k. \tag{5}$$

Suppose some Egorov hydrodynamic type system has the local Hamiltonian structure

$$c_t^i = \partial_x \left(\bar{g}^{ik} \frac{\partial \mathbf{h}}{\partial c^k} \right), \qquad i = 1, 2, ..., N, \tag{6}$$

where $\bar{g}^{ik}$ is a constant symmetric non-degenerate matrix (see details in [10]). Then, a corresponding solution of the WDVV equation can be found (see [1], [9], [24]; and also [5], [11]). The first publication [9] describing this connection of the Egorov hydrodynamic type systems and the WDVV equation was appear 14 years ago, but a deficit of explicit solutions still exists.

In this paper we present an *effective algorithm* allowing to construct infinitely many particular solutions of the WDVV equation written in an *explicit form* via flat coordinates a^k of corresponding Egorov hydrodynamic type systems. These hydrodynamic

type systems are Hamiltonian hydrodynamic reductions (6) of the Egorov hydrodynamic chains (see [33]). Thus, in this paper we show how these hydrodynamic reductions can be converted in solutions of the WDVV equation.

Let us emphasize again: any solution of the WDVV equation determines an integrable hierarchy of the Egorov hydrodynamic type systems; each Egorov Hamiltonian hydrodynamic type system determines a solution of the WDVV equation. It means that a description of all Egorov Hamiltonian hydrodynamic type systems is equivalent to a description of all solutions of the WDVV equation. Thus, in this paper we are able to present a large list of new solutions of the WDVV equation written in an explicit form via flat coordinates.

The **scheme**: The algorithm presented in this paper contains following six steps:

1. Suppose the **Egorov** Hamiltonian hydrodynamic type system (6) is given via the Riemann invariants (3). Then the Lame coefficients H_k can be found from (5), and the rotation coefficients β_{ik} of the corresponding *conjugate* curvilinear coordinate net are given by (see [7])

$$\beta_{ik} = \frac{\partial_i H_k}{H_i}, \qquad i \neq k.$$

2. These rotation coefficients β_{ik} satisfy the Bianchi–Darboux–Lame–**Egorov** system

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \qquad i \neq j \neq k, \qquad\qquad \beta_{jk} = \beta_{kj}. \tag{7}$$

The linear PDE system

$$\partial_i \tilde{H}_k = \beta_{ik}\tilde{H}_i, \qquad i \neq k \tag{8}$$

has a general solution parameterized by N arbitrary functions of a single variable.

The existence of the local Hamiltonian structure (6) is equivalent the zero curvature condition (see [7])

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi}\beta_{mk} = 0, \tag{9}$$

which is compatible with (7). Thus, the rotation coefficients β_{ik} describe corresponding *orthogonal* curvilinear coordinate net.

It means that the linear PDE system (8) can be replaced by N ODE systems (see [42])

$$\partial_i H_k^{(n,s)} = \beta_{ik} H_i^{(n,s)}, \;\; i \neq k; \qquad \partial_i H_i^{(n,s)} + \sum_{m \neq i} \beta_{mi} H_m^{(n)} = H_m^{(n-1,s)}, \quad n = 0, 1, ...,$$

where $H_m^{(0,s)}$ are solutions of N ODE systems

$$\partial_i H_k^{(0,s)} = \beta_{ik} H_i^{(0,s)}, \quad i \neq k; \qquad \partial_i H_i^{(0,s)} + \sum_{m \neq i} \beta_{mi} H_m^{(0,s)} = 0, \quad i, k, s = 1, 2, ..., N. \tag{10}$$

Definition: *The Lame coefficients* $\bar{H}_k^{(s)} \equiv H_k^{(0,s)}$ *are said to be* ***canonical*** *Lame coefficients.*

3. Let us introduce the **constant** matrix

$$\bar{g}^{ik} = \sum \bar{H}_m^{(i)} \bar{H}_m^{(k)}$$

and the canonical Lame coefficients with sub-indexes $\bar{H}_{(s)k} = \bar{g}_{sm}\bar{H}_k^{(m)}$.

Let us choose any solution of (10) as $\bar{H}_{(1)k}$. Then let us construct $N-1$ commuting flows

$$r^i_{t^k} = \frac{\bar{H}_{(k)i}}{\bar{H}_{(1)i}} r^i_{t^1}, \qquad i = 1, 2, ..., N, \qquad k = 2, 3, ..., N, \tag{11}$$

where all Riemann invariants r^k are functions of $t^1, t^2, ..., t^N$.

4. Let us introduce $N(N+1)/2$ functions $a_{kn}(\mathbf{r})$ such that

$$\partial_i a_{kn} = \bar{H}_{(k)i}\bar{H}_{(n)i}. \tag{12}$$

Definition: *The functions a_{k1} are said to be the* ***adjoint*** *flat coordinates a_k.*

Then all other functions a_{kn} can be expressed via these field variables a_k, because they are connected with the Riemann invariants r^n by an invertible transformation $a_k(\mathbf{r})$. One can differentiate adjoint flat coordinates a_i with respect to times t^k. Then the hydrodynamic type systems (11) written via the Riemann invariants r^n can be written in the symmetric (see (12)) conservative form

$$\partial_{t^k} a_i = \partial_{t^1} a_{ik}(\mathbf{a}), \qquad i, k = 1, 2, ..., N. \tag{13}$$

Definition [34]: *The hydrodynamic type systems (13) are said to be the* ***canonical Egorov basic set****.*

Remark: Since the functions $a_{ik}(\mathbf{a})$ are symmetric (see (12)), then the unique function $\Omega(t^1, t^2, ..., t^N)$ can be introduced and determined by its second derivatives

$$a_{ik}(\mathbf{a}) = \frac{\partial^2\Omega}{\partial t^i \partial t^k}, \qquad i, k = 1, 2, ..., N. \tag{14}$$

Thus, the hydrodynamic type systems (13) can be replaced by $N(N-1)/2$ algebraic equations

$$\frac{\partial^2\Omega}{\partial t^i \partial t^k} = a_{ik}\left(\frac{\partial^2\Omega}{\partial t^j \partial t^1}\right), \qquad i, k = 2, ..., N. \tag{15}$$

5. Since the hydrodynamic type system (3) possesses the Hamiltonian structure (6), then its any commuting flow

$$r^i_\tau = w^i(\mathbf{r}) r^i_x, \qquad i = 1, 2, ..., N$$

possesses the same Hamiltonian structure

$$c^i_\tau = \partial_x\left(\bar{g}^{is}\frac{\partial \bar{\mathbf{h}}}{\partial c^s}\right), \qquad i, k = 1, 2, ..., N.$$

Since the hydrodynamic type system (3) possesses the Egorov pair (4), then its any commuting flow possesses the similar Egorov pair (see [38])

$$a_\tau = h_x, \qquad h_\tau = q_x. \tag{16}$$

Lemma: *N auxiliary commuting flows* (cf. (11))

$$r^i_{t^k} = \frac{\bar{H}_{(k)i}}{H_i} r^i_x, \qquad i, k = 1, 2, ..., N \tag{17}$$

are determined by the extra conservation law (4)

$$a_{t^k} = \partial_x c_k, \tag{18}$$

where a is a potential of the Egorov metric.

Proof: Indeed, if (18) is the conservation law of (17), then $\partial_i c_k = \bar{H}_{(k)i} H_i$. Thus, the adjoint flat coordinates c_k determine auxiliary N commuting flows (17).

Let us rewrite these N hydrodynamic type systems in the potential form

$$d\xi^i = c^i dx + \bar{g}^{is} \frac{\partial \mathbf{h}_k}{\partial c^s} dt^k. \tag{19}$$

Since the local Hamiltonian structure preserves under arbitrary linear change of independent variables (x, t^k), then one can introduce a new set of flat coordinates (see [37])

$$a^k = \bar{g}^{ks} \frac{\partial \mathbf{h}_1}{\partial c^s}, \tag{20}$$

then (19)

$$d\tilde{\xi}^i = \bar{g}^{is} \frac{\partial \mathbf{h}_1}{\partial c^s} dt^1 + \bar{g}^{is} \frac{\partial \mathbf{h}_k}{\partial c^s} dt^k = a^i dt^1 + \bar{g}^{is} \frac{\partial \tilde{\mathbf{h}}_k}{\partial a^s} dt^k \tag{21}$$

leads to the

Lemma [9]: *The Hamiltonian densities $\tilde{\mathbf{h}}_k$ are determined by the unique function $F(a^1, a^2, ..., a^N)$*

$$\tilde{\mathbf{h}}_k = \frac{\partial F}{\partial a^k}. \tag{22}$$

Proof: $N-1$ commuting flows (13) possess the local Hamiltonian structure (see (21))

$$\partial_{t^k}(\bar{g}_{ns} a^s) = \partial_{t^1} \frac{\partial \tilde{\mathbf{h}}_k}{\partial a^n}. \tag{23}$$

If to substitute $k = 1$ in (23), then

$$\bar{g}_{is} a^s = \frac{\partial \tilde{\mathbf{h}}_1}{\partial a^i}. \tag{24}$$

If to substitute $n = 1$ in (23) and to take into account that by definition $a_{i1} \equiv a_i$ (see (12)), then

$$\frac{\partial \tilde{\mathbf{h}}_k}{\partial a^1} = a_k. \tag{25}$$

The comparison (25) with (24) implies the existence of some potential function $F(a^1, a^2, ..., a^N)$ (see (22)).

6. Thus, these $N-1$ commuting flows (23) can be written in the form

$$a^i_{t^k} = \partial_{t^1} \left(\bar{g}^{is} \frac{\partial^2 F}{\partial a^s \partial a^k} \right). \tag{26}$$

Corollary [9]: Since (see (22), (24) and (25))

$$\bar{g}_{is}a^s = \frac{\partial^2 F}{\partial a^i \partial a^1},$$

then (with the aid of a differentiation)

$$\frac{\partial^3 F}{\partial a^i \partial a^k \partial a^1} = \bar{g}_{ik}$$

and (with the aid of an integration)

$$\tilde{\mathbf{h}}_1 = \frac{\partial F}{\partial a^1} = \frac{1}{2}\bar{g}_{sk}a^s a^k. \tag{27}$$

This is the first nontrivial conservation law density (except the flat coordinates a^k and the Hamiltonian densities $\tilde{\mathbf{h}}_k$ $(k = 2, 3, ..., N)$. This is a momentum density, and a corresponding conservation law is given by

$$\partial_{t^k}\tilde{\mathbf{h}}_1 = \partial_{t^1}\left(a^s \frac{\partial \tilde{\mathbf{h}}_k}{\partial a^s} - \tilde{\mathbf{h}}_k\right).$$

Corollary [9]: In general case $\bar{g}_{11} \neq 0$, then (see (27)) the function F is *cubic* with respect to a^1, i.e.

$$F = \frac{1}{6}\bar{g}_{11}(a^1)^3 + \frac{1}{2}\sum_{k>1}\bar{g}_{1k}a^k(a^1)^2 + \frac{1}{2}\sum_{k,s>1}\bar{g}_{sk}a^s a^k a^1 + \Psi(a^2, a^3, ..., a^N),$$

where the function $\Psi(a^2, a^3, ..., a^N)$ can be determined from the compatibility conditions

$$\partial_{t^k}(\partial_{t^n}a_i) = \partial_{t^n}(\partial_{t^k}a_i), \qquad i, k, n = 2, 3, ..., N. \tag{28}$$

Theorem [9]: *The above function $F(a^1, a^2, ..., a^N)$ satisfies the WDVV equation*

$$\frac{\partial^3 F}{\partial a^k \partial a^i \partial a^s}\bar{g}^{sp}\frac{\partial^3 F}{\partial a^p \partial a^j \partial a^n} = \frac{\partial^3 F}{\partial a^j \partial a^i \partial a^s}\bar{g}^{sp}\frac{\partial^3 F}{\partial a^p \partial a^k \partial a^n}. \tag{29}$$

Proof: The substitution (26) in (28) yields the WDVV equation.
Observation: Let us compare (13), (14) and (26). Then we obtain

$$\frac{\partial^2 \Omega}{\partial t^k \partial t^n} = \frac{\partial^2 F}{\partial a^k \partial a^n}. \tag{30}$$

Thus, *any N component hydrodynamic reduction* (13) *of multi-dimensional nonlinear equation (cf.* (15))

$$Q\left(\frac{\partial^2 \Omega}{\partial t^k \partial t^n}\right) = 0 \tag{31}$$

can be interpreted as some solution of the WDVV equation (29). For instance, the dispersionless limit of the KP equation (known also as Khohlov–Zabolotzkaya equation)

$$\frac{\partial^2 \Omega}{\partial t \partial t} = \frac{\partial^2 \Omega}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial^2 \Omega}{\partial x^2}\right)^2$$

possesses some solutions of three component WDVV equation (29) with the extra constraint

$$\frac{\partial^2 F}{\partial b \partial b} = \frac{\partial^2 F}{\partial a \partial c} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial a^2}\right)^2,$$

where a, b, c are flat coordinates.

The Heavenly equation (see, for instance, [18])

$$\frac{\partial^2 \Omega}{\partial x \partial y}\frac{\partial^2 \Omega}{\partial z \partial t} - \frac{\partial^2 \Omega}{\partial x \partial z}\frac{\partial^2 \Omega}{\partial y \partial t} = 1$$

possesses some solutions of four component WDVV equation (29) with the extra constraint

$$\frac{\partial^2 F}{\partial a \partial c}\frac{\partial^2 F}{\partial u \partial b} - \frac{\partial^2 F}{\partial a \partial u}\frac{\partial^2 F}{\partial c \partial b} = 1,$$

where a, b, c, u are flat coordinates.

In this paper we construct solutions of the WDVV equation **avoiding the Riemann invariants** r^k (see (3)). We start from the Egorov hydrodynamic type system written in the conservative form (6) and construct the canonical Egorov basic set (26). Then a corresponding solution of the WDVV equation can be found in quadratures.

Let us introduce the unique function Ω depended on infinitely many independent variables x^k ($k = 0, \pm 1, \pm 2, \pm 3, ...$). Let us introduce new functions

$$\mathbf{H}_k = \frac{\partial^2 \Omega}{\partial x^0 \partial x^k}. \tag{32}$$

Suppose all other derivatives $\partial^2 \Omega / \partial x^n \partial x^k$ can be expressed via $\mathbf{H}_k$, i.e. (cf. (31))

$$\frac{\partial^2 \Omega}{\partial x^n \partial x^k} = \Phi_{kn}(\mathbf{H}).$$

Let us introduce following rule [33]: if $k, n > 0$, then $\Phi_{kn}(\mathbf{H}) = \Phi_{kn}(\mathbf{H}_0, \mathbf{H}_1, ..., \mathbf{H}_{k+n})$, if $k > 0$, but $n < 0$, then $n = -m$ and $\Phi_{-m,k}(\mathbf{H}) = \Phi_{-m,k}(\mathbf{H}_{-m}, ..., \mathbf{H}_{-1}, \mathbf{H}_0, \mathbf{H}_1, ..., \mathbf{H}_k)$, if $k, n < 0$, then $\Phi_{-k,-n}(\mathbf{H}) = \Phi_{-k,-n}(\mathbf{H}_{-k-n}, ..., \mathbf{H}_{-1}, \mathbf{H}_0)$.

Definition [33]: *The family of hydrodynamic chains*

$$\partial_{x^n} \mathbf{H}_k = \partial_{x^0} \Phi_{kn}(\mathbf{H})$$

*is said to be **Egorov** hydrodynamic chains.*

Suppose all moments $\mathbf{H}_k$ are functions of N Riemann invariants only (see [15]). These functions must be compatible with the Egorov hydrodynamic chain for all N. Suppose all such functions are found. It means, that infinitely many solutions (of a corresponding Egorov hydrodynamic chain) parameterized by N arbitrary functions of a single variable are found too. Suppose we are able to extract all hydrodynamic reductions (3) determined by the zero curvature condition (9). It means, that we are able to construct a corresponding solution of the WDVV equation. This is a goal of this paper. Without lost of generality we restrict our consideration on well-known integrable hydrodynamic chains

(1) and (2). These hydrodynamic chains possess infinitely many local Hamiltonian structures (see [32]). Thus, infinitely many Egorov Hamiltonian hydrodynamic reductions can be converted in corresponding solutions of the WDVV equation.

Remark: We use different notation for "hydrodynamic type system times" t^k and for "hydrodynamic chain times" x^n to emphasize that in general case these independent variables **do not coincide**.

Egorov (external) criterion [33]: If the hydrodynamic chain

$$A_t^k = \sum_{n=0}^{k+1} F_n^k(\mathbf{A}) A_x^n, \qquad k = 0, 1, 2, ...$$

has infinitely many N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable and the couple of conservation laws (4), then the function Ω can be determined via an appropriate set of conservation law densities (32).

Example [34]: The Benney hydrodynamic type system has infinitely many N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable and the couple of conservation laws (4)

$$\partial_t A^0 = \partial_x A^1, \qquad \partial_t A^1 = \partial_x \left(A^2 + \frac{(A^0)^2}{2} \right). \tag{33}$$

Thus, A^0 is a potential of the Egorov metric for any hydrodynamic reduction of the Benney hydrodynamic chain.

The paper is organized in the following order. In the second section all "flat" hydrodynamic reductions of the Benney–DKP moment chain are described. In the third section a new solution of the WDVV equation associated with the so-called waterbag reduction is constructed. In the fourth section a link between solutions of the WDVV equation and some degenerations of the waterbag reduction are given. In the fifth section new solution of the WDVV equation is derived from corresponding reduction of the modified Benney hydrodynamic chain. In the sixth section new solutions of the WDVV equations associated with corresponding hydrodynamic reductions of the dBKP/Veselov–Novikov hierarchy are found.

2 Benney hydrodynamic chain

One of the most interesting questions of the classical differential geometry which has appeared at studying of semi-Hamiltonian systems of hydrodynamic type is the description of the surfaces admitting non-trivial deformations with preservation of principal directions and principal curvatures. Then the number of essential parameters on which such deformations depend, is actually equal to number of various local Hamiltonian structures of corresponding system of hydrodynamic type [13]. Such local Hamiltonian structures are determined by a differential-geometric Poisson bracket of the first order [10]. It has been proved that $N-$component hydrodynamic type system cannot possess more than $(N+1)$ local Hamiltonian structures of Dubrovin-Novikov type [17]. Integrable hydrodynamic chains (see [36], [33], [12]) are infinitely many component generalizations of

semi-Hamiltonian hydrodynamic type systems. Thus, formally, an existence of infinitely many local Hamiltonian structures is not forbidden. Indeed, the Kupershmidt hydrodynamic chains possess infinitely many local Hamiltonian structures (see [32]). Since the modified Benney hydrodynamic chain is a particular case of the Kupershmidt hydrodynamic chain (2), then the Benney hydrodynamic chain also has infinite many local Hamiltonian structures, then infinitely many Egorov Hamiltonian hydrodynamic reductions can be extracted. Then corresponding solutions of the WDVV equation can be obtained in an explicit form.

The first example of integrable hydrodynamic chains was found in the theory describing dynamics of a finite depth fluid (see (1) and [3]).

Theorem [19]: *A deformation of the Riemann mapping*

$$\lambda = \mu + \frac{A^0}{\mu} + \frac{A^1}{\mu^2} + \frac{A^2}{\mu^3} + ... \tag{34}$$

describes by the Gibbons equation

$$\lambda_t - \mu\lambda_x = \frac{\partial\lambda}{\partial\mu}\left[\mu_t - \partial_x\left(\frac{\mu^2}{2} + A^0\right)\right], \tag{35}$$

where a dynamics of the moments A^k is given by (1).

In this section we describe *truncations* of the Riemann mapping according to different moment decompositions $A^k(\mathbf{r})$ (see (3)). Such truncations of the Riemann mapping we call the *Riemann surfaces*. We are interested in such moment decompositions $A^k(\mathbf{r})$, that corresponding Egorov hydrodynamic type systems can be recognized as the Hamiltonian hydrodynamic reductions (6).

The Gibbons equation has plenty distinguished features. We mark two of them (see [19]).

1. $\lambda =$ const, then $\partial\lambda/\partial\mu \neq 0$ and the Gibbons equation reduces to the generating function of conservation laws

$$\mu_t = \partial_x\left(\frac{\mu^2}{2} + A^0\right). \tag{36}$$

The substitution of the inverse series (34)

$$\mu = \lambda - \frac{\mathbf{H}_0}{\lambda} - \frac{\mathbf{H}_1}{\lambda^2} - \frac{\mathbf{H}_2}{\lambda^3} - ...$$

in (36) yields an infinite series of the Kruskal conservation laws

$$\partial_t\mathbf{H}_k = \partial_x[\mathbf{H}_{k+1} - \frac{1}{2}\sum_{m=0}^{k-1}\mathbf{H}_m\mathbf{H}_{k-1-m}], \qquad k = 0, 1, 2, ..., \tag{37}$$

where conservation law densities $\mathbf{H}_k$ are polynomial functions of A^n. For instance, $\mathbf{H}_0 = A^0$, $\mathbf{H}_1 = A^1$, $\mathbf{H}_2 = A^2 + (A^0)^2$, $\mathbf{H}_3 = A^3 + 3A^0A^1$.

Remark: The Benney hydrodynamic chain has an infinite series of conservation law densities $\mathbf{H}_k$. Since in this paper we consider hydrodynamic reductions of the Benney

hydrodynamic chain, we shall use the special notation $\mathbf{h}_k(\mathbf{r})$ for corresponding *reduced* conservation law densities $\mathbf{H}_k$.

2. The condition $\partial\lambda/\partial\mu = 0$ determines the branch points of the Riemann surface $r^i = \partial\lambda/\partial\mu|_{\mu=\mu^i}$, then the Gibbons equation reduces to the hydrodynamic type system (3)

$$r_t^i = \mu^i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N, \tag{38}$$

also can be written in the conservative form (see (36); $\mu \to c^i$)

$$c_t^i = \partial_x \left(\frac{(c^i)^2}{2} + A^0(\mathbf{c}) \right), \tag{39}$$

where the moment A^0 is a solution of the Gibbons–Tsarev system (see [35])

$$\begin{aligned} (c^i - c^k)\partial_{ik}A^0 &= \partial_k A^0 \partial_i \left(\sum \partial_n A^0 \right) - \partial_i A^0 \partial_k \left(\sum \partial_n A^0 \right), \quad i \neq k, \\ (c^i - c^k)\frac{\partial_{ik}A^0}{\partial_i A^0 \partial_k A^0} &+ (c^k - c^j)\frac{\partial_{jk}A^0}{\partial_j A^0 \partial_k A^0} + (c^j - c^i)\frac{\partial_{ij}A^0}{\partial_i A^0 \partial_j A^0} = 0, \quad i \neq j \neq k. \end{aligned} \tag{40}$$

However, this is not an unique conservative form. For instance, the hydrodynamic type system (38) can be written in the form (see (1) and (39))

$$\begin{aligned} A_t^k &= A_x^{k+1} + kA^{k-1}A_x^0, \quad k = 0, 1, ..., K-2, \qquad A_t^{K-1} = \partial_x A^K(\mathbf{c}) + (K-1)A^{K-2}A_x^0, \\ c_t^i &= \partial_x \left(\frac{(c^i)^2}{2} + A^0 \right), \quad i = 1, 2, ..., M, \end{aligned}$$

where $A^K(\mathbf{c})$ can be found from the consistency of the above hydrodynamic type system and its generating function of conservation laws (36) (cf. (40), where $K = 0$).

Main theorem [35]: *the most general* ***Hamiltonian*** *hydrodynamic reduction* (6)

$$c_t^i = \frac{1}{\varepsilon_i}\partial_x \frac{\partial \mathbf{h}_{K+2}}{\partial c^i}, \qquad \partial_t h_k = \partial_x \frac{\partial \mathbf{h}_{K+2}}{\partial h_{K-1-k}}, \quad k = 0, 1, 2, ..., K-1 \tag{41}$$

is connected with the equation of the Riemann surface

$$\lambda = \frac{\mu^{K+1}}{K+1} + \sum_{k=0}^{K-1} Q_{K-1-k}(\mathbf{A})\mu^k - \sum_{m=1}^{M} \varepsilon_m \ln(\mu - c^m), \tag{42}$$

where ε_k are arbitrary constants, $h_k(\mathbf{A})$ and $Q_k(\mathbf{A})$ are some polynomials with respect to the moments A^n.

If $\Sigma\varepsilon_k = 0$, then $Q_k(\mathbf{A})$ can be obtained by a comparison of the above expression (6) with (34)

$$\frac{\mu^{K+1}}{K+1} + \sum_{k=0}^{K-1} Q_{K-1-k}(\mathbf{A})\mu^k - \sum_{m=1}^{M} \varepsilon_m \ln(\mu - c^m) = \frac{1}{K+1} \left(\mu + \sum_{k=0}^{\infty} \frac{A^k}{\mu^{k+1}} \right)^{K+1}.$$

Then all higher moments A^k are given by the moment decomposition

$$A^k = \frac{1}{k+1}\sum_{m=1}^{M}\varepsilon_m (c^m)^{k+1}, \qquad k = N, N+1, ...$$

If $\Sigma\varepsilon_k \neq 0$, then at first the parameter λ must be replaced by the combination $\lambda(\mu) \to \lambda(\mu) - \Sigma\varepsilon_k \ln \lambda(\mu)$.

The Hamiltonian hydrodynamic type system (41) (was found by L.V. Bogdanov and B.G. Konopelchenko in [4]) has two *most degenerate* limits: All constants $\varepsilon_k = 0$. Thus, this is nothing but the Lax hydrodynamic reduction (see [9] and [24], the dispersionless limit of the Gelfand–Dikey linear problem). This hydrodynamic reduction can be obtained by the sole constraint $\mathbf{h}_K = 0$, then $\mathbf{h}_{K+1} = \Sigma h_k h_{K-1-k}/2$ (see (37)) is the momentum density.

The second case is the Kodama hydrodynamic reduction (see [22]) associated with the equation of the Riemann surface

$$\lambda = \frac{\mu^{K+1}}{K+1} + \sum_{k=0}^{K-1} Q_{K-1-k}(\mathbf{A})\mu^k + \sum_{n=1}^{L}\frac{b_n}{(\mu - c)^n}, \tag{43}$$

which is sub-case of the Krichever hydrodynamic reduction (see [24], and also [1]) associated with the equation of the Riemann surface

$$\lambda = \frac{\mu^{K+1}}{K+1} + \sum_{k=0}^{K-1} Q_{K-1-k}(\mathbf{A})\mu^k + \sum_{m=1}^{M}\sum_{n=1}^{N_m}\frac{b_{m,n}}{(\mu - c^m)^n}. \tag{44}$$

3 The waterbag reduction and the WDVV equation

Let us consider the hydrodynamic reduction (41), where $K = 0$ and M is arbitrary.

This is the so-called waterbag hydrodynamic reduction (see [21], [22])

$$c_t^i = \partial_x\left(\frac{(c^i)^2}{2} + \sum \varepsilon_k c^k\right). \tag{45}$$

In such a case the equation of the Riemann surface is given by (42)

$$\lambda = \mu - \sum_{m=1}^{M}\varepsilon_m \ln(\mu - c^m). \tag{46}$$

In this paper for simplicity we restrict our consideration on the case $\Sigma\varepsilon_k = 0$ (if $\Sigma\varepsilon_k \neq 0$, then corresponding computation became more complicated, but the approach presented below remains valid).

1. Since $\lambda(x,t)$ is a solution of the linear equation (35), then λ can be replaced by any function $\tilde{\lambda}(\lambda)$. Thus, (46) can be written in the form

$$\lambda = (\mu - c^i)e^{-\mu/\varepsilon_i}\prod_{k\neq i}(\mu - c^k)^{\varepsilon_k/\varepsilon_i}$$

for any fixed index i. Then N infinite series of conservation laws

$$\mu^{(i)} = c^i + h_i^{(1)}(\mathbf{c})\lambda + h_i^{(2)}(\mathbf{c})\lambda^2 + h_i^{(3)}(\mathbf{c})\lambda^3 + ... \tag{47}$$

can be obtained with the aid of the Bürmann–Lagrange series (see [27]), whose coefficients are determined by

$$h_i^{(n)} = \frac{1}{n!}\frac{d^{n-1}}{d(c^i)^{n-1}}\left(e^{nc^i/\varepsilon_i}\prod_{k\neq i}(c^i - c^k)^{-n\varepsilon_k/\varepsilon_i}\right), \qquad n = 1, 2, ...$$

For instance, the first conservation law densities are

$$h_i^{(1)} = e^{c^i/\varepsilon_i}\prod_{k\neq i}(c^i - c^k)^{-\varepsilon_k/\varepsilon_i}, \quad h_i^{(2)} = \frac{e^{2c^i/\varepsilon_i}}{\varepsilon_i}\left(1 - \sum_{n\neq i}\frac{\varepsilon_n}{c^i - c^n}\right)\prod_{k\neq i}(c^i - c^k)^{-2\varepsilon_k/\varepsilon_i}, ...$$

All above conservation law densities are found by a differentiation only. Below we develop the approach successfully applied to the Zakharov hydrodynamic reductions of the Benney hydrodynamic chain (see [39]) for construction of conservation law densities for the Egorov Hamiltonian hydrodynamic type systems.

2. The generating function of conservation laws (36) is consistent with any hydrodynamic reduction written via the Riemann coordinates (see (3))

$$r_t^i = \mu^i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N. \tag{48}$$

It means that the generating function of conservation law densities μ satisfies the nonlinear PDE system of the first order

$$\partial_i\mu = \frac{\partial_i A^0}{\mu^i - \mu}. \tag{49}$$

At the same time, the moment $A^0(\mathbf{r})$ is a potential of the Egorov metric (see (33) and [34]). Then an arbitrary commuting flow

$$r_\tau^i = w^i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N \tag{50}$$

has the corresponding conservation law $A_\tau^0 = h_x$, where $h(\mathbf{r})$ is a conservation law density.

Lemma: *The generating function of commuting flows* (50)

$$r_{\tau(\zeta)}^i = \frac{1}{\mu^i - \mu}r_x^i, \qquad i = 1, 2, ..., N \tag{51}$$

is connected with the conservation law $\partial_{\tau(\zeta)}A^0 = \mu_x$. *The generating function of conservation laws and commuting flows is given by*

$$\partial_{\tau(\zeta)}\mu(\lambda) = \partial_x \ln[\mu(\lambda) - \mu(\zeta)]. \tag{52}$$

Proof: Let us differentiate the conservation law $A_{\tau(\zeta)}^0 = \partial_x\mu(\zeta)$ with respect to the Riemann invariants. Then (see (50))

$$w^i = \frac{\partial_i\mu}{\partial_i A^0},$$

and (51) can be obtained taking into account (49). Differentiation of the generating function (52) with respect to the Riemann invariants yields the same result (51).

3. N *primary* commuting flows in the Riemann invariants (see (47))

$$r^i_{t^k} = \frac{1}{\mu^i - c^k} r^i_x, \qquad i = 1, 2, ..., N \tag{53}$$

can be obtained directly from the generating function (51) according N punctures c^i, where $\mu(\zeta) \to c^k$ and $\tau(\zeta) \to t^k$ (see (46)). These commuting flows in the conservative form are

$$\mu_{t^k} = \partial_x \ln(\mu - c^k).$$

Another choice of punctures c^i, where $\mu(\lambda) \to c^i$ leads to the generating function of commuting flows

$$c^i_\tau = \partial_x \ln(c^i - \mu), \tag{54}$$

while the hydrodynamic type system (48) in the conservative form is given by (45).

4. Since the waterbag reduction (45) is the Hamiltonian hydrodynamic type system

$$c^k_t = \frac{1}{\varepsilon_k} \partial_x \frac{\partial \mathbf{h}_2}{\partial c^k}, \tag{55}$$

where $\mathbf{h}_2 = A^2 + (A^0)^2$, then each commuting flow has the same local Hamiltonian structure

$$c^k_{t^s_p} = \frac{1}{\varepsilon_k} \partial_x \frac{\partial \mathbf{h}^{(p)}_s}{\partial c^k}, \qquad k, s = 1, 2, ..., N, \quad p = 0, 1, 2, ... \tag{56}$$

Lemma: *The generating function of commuting flows* (54) *is determined by the Hamiltonian density*

$$\mathbf{h}(\zeta) = \frac{\mu^2(\zeta)}{2} + A^0 - \sum \varepsilon_k c^k \ln[\mu(\zeta) - c^k]. \tag{57}$$

Proof: Since the generating function of commuting flows (54) has the same Hamiltonian structure (55), then an integration of the differential (see (54) and (55))

$$d\mathbf{h}(\zeta) = \sum \varepsilon_k \ln[\mu(\zeta) - c^k] dc^k$$

yields (57).

The term A^0 in the Hamiltonian density (57) is inessential. By this reason we remove it from a further consideration. At the beginning of this section we mentioned the existence on $N+2$ infinite series of conservation laws. The substitution of the Kruskal series in (57) yields the same set of the Kruskal conservation laws. However, the substitution of the Bürmann–Lagrange series (47) at the first step yields the logaritmic conservation law densities

$$\tilde{h}^{(1)}_i = \frac{(c^i)^2}{2} + \sum_{k \neq i} \varepsilon_k (c^k - c^i) \ln(c^k - c^i). \tag{58}$$

The corresponding hydrodynamic type systems are (53).

5. $A^0 = \Sigma\varepsilon_k c^k$ is a potential of the Egorov metric (see (33), (55) and (56))

$$A^0_t = \partial_x(\delta\mathbf{h}_2) = A^1_x, \qquad A^0_{t^s_p} = \partial_x(\delta\tilde{h}^{(p)}_s) = \partial_x\tilde{h}^{(p-1)}_s,$$

where $\delta = \Sigma\partial/\partial c^k$ is a shift operator and $\tilde{h}^{(0)}_i = c^i$. Since $A^0_{\tau(\zeta)} = \mu_x$, then it means that $\delta\mathbf{h}_{p+1} = \mathbf{h}_p$, $\delta h^{(p)}_s = h^{(p-1)}_s$, $p = 0, 1, 2, ...,$ $s = 1, 2, ..., N$.

Lemma: *N commuting flows* (17)

$$c^i_{t^k} = \frac{1}{\varepsilon_i}\partial_x\frac{\partial\tilde{h}^{(1)}_k}{\partial c^i} \tag{59}$$

are determined by the conservation laws

$$A^0_{t^k} = \partial_x c^k,$$

where $t^k_1 \equiv t^k$.

Proof: Indeed, $\delta\tilde{h}^{(1)}_k = \tilde{h}^{(0)}_k = c^k$ (see (58)).

Let us write the above commuting flows (59) in the potential form (19)

$$d\xi^i = c^i dx + \sum_{k\neq i}\ln(c^i - c^k)dt^k + \frac{1}{\varepsilon_i}\left(c^i - \sum_{k\neq i}\ln(c^i - c^k)\right)dt^i.$$

Let us introduce new flat coordinates (20)

$$a_1 = \frac{1}{\varepsilon_1}\left(c^1 - \sum_{k\neq 1}\varepsilon_k\ln(c^1 - c^k)\right), \qquad a_k = \ln(c^1 - c^k), \quad k = 2, 3, ..., N.$$

Since, the above point transformation is invertible

$$c^1 = \sum\varepsilon_n a_n, \qquad c^k = \sum\varepsilon_n a_n - e^{a_k}, \quad k = 2, 3, ..., N$$

the canonical Egorov basic set is given by

$$\begin{aligned} \partial_{t^n}a_1 &= \partial_{t^1}a_n, \qquad \partial_{t^n}a_k = \partial_{t^1}\ln(e^{a_k} - e^{a_n}), \qquad k \neq 1, n, \\ \partial_{t^n}a_n &= \frac{1}{\varepsilon_n}\partial_{t^1}\left(\sum\varepsilon_m a_m - \varepsilon_1 a_n - e^{a_n} - \sum_{m\neq 1,n}\varepsilon_m\ln(e^{a_m} - e^{a_n})\right). \end{aligned} \tag{60}$$

Thus, the above hydrodynamic type systems can be written in the form (26), where $a^k = \varepsilon_k a_k$. Then the corresponding solution of the WDVV equation (29)

$$\sum\frac{1}{\varepsilon_s}\left(\frac{\partial^3 F}{\partial a_k\partial a_i\partial a_s}\frac{\partial^3 F}{\partial a_s\partial a_j\partial a_n} - \frac{\partial^3 F}{\partial a_j\partial a_i\partial a_s}\frac{\partial^3 F}{\partial a_s\partial a_k\partial a_n}\right) = 0 \tag{61}$$

can be found in quadratures

$$F = \frac{\varepsilon_1^2(a_1)^3}{6} + \frac{\varepsilon_1 a_1}{2}\sum\varepsilon_m(a_m)^2 + P_3(\mathbf{a}) - \sum\varepsilon_m e^{a_m} + \frac{1}{2}\sum_{m<k}\varepsilon_k\varepsilon_m\left[\mathrm{Li}_3\left(e^{a_k - a_m}\right) + \mathrm{Li}_3\left(e^{a_m - a_k}\right)\right],$$

where

$$P_3(\mathbf{a}) = \sum \frac{\varepsilon_m(\varepsilon_m - \varepsilon_1)(a_m)^3}{6} + \sum_{m<k} \frac{\varepsilon_k\varepsilon_m}{12}\left[(a_k + a_m)^3 - 2((a_k)^3 + (a_m)^3)\right].$$

Attention: the index 1 **is not included** in the above two summations.

For instance, if $N = 3$, then

$$\begin{aligned} F &= \frac{\varepsilon_1^2(a_1)^3}{6} + \frac{\varepsilon_1 a_1}{2}[\varepsilon_2(a_2)^2 + \varepsilon_3(a_3)^2] + \frac{\varepsilon_2\varepsilon_3}{4}[a_3(a_2)^2 + a_2(a_3)^2] \\ &+ \left(\frac{\varepsilon_2(\varepsilon_2 - \varepsilon_1)}{6} - \frac{\varepsilon_2\varepsilon_3}{12}\right)(a_2)^3 + \left(\frac{\varepsilon_3(\varepsilon_3 - \varepsilon_1)}{6} - \frac{\varepsilon_2\varepsilon_3}{12}\right)(a_3)^3 \\ &- \varepsilon_2 e^{a_2} - \varepsilon_3 e^{a_3} + \frac{\varepsilon_2\varepsilon_3}{2}\left[\mathrm{Li}_3\left(e^{a_3 - a_2}\right) + \mathrm{Li}_3\left(e^{a_2 - a_3}\right)\right]. \end{aligned} \tag{62}$$

Remark: Similar (but different!) solutions are found by R. Martini and L.K. Hoevenaars in [28].

4 Degenerations of the waterbag hydrodynamic reduction

The waterbag hydrodynamic reduction is a most general Hamiltonian hydrodynamic reduction (6), which can be obtained by the *Dirac restriction* (see [16]) of the Kupershmidt–Manin Poisson bracket for the Benney hydrodynamic chain (see [35]). All other "flat" hydrodynamic reductions connected with the equation of the Riemann surface

$$\lambda = \mu - \sum_{k=1}^{N_0} \varepsilon_k \ln(\mu - a_0^k) + \sum_{m=1}^{M}\sum_{k=1}^{N_m} \frac{\tilde{a}_m^k}{(\mu - a_m^k)^m}$$

and obtained by the Dirac restriction of the Kupershmidt–Manin Poisson bracket can be derived from the waterbag hydrodynamic reduction by an appropriate degeneration (see [35]). In this case M is an arbitrary integer (the number of equations in corresponding hydrodynamic reduction is $N = \sum_{m=0}^{M}(m+1)N_m$, where N_m are arbitrary integers). For instance, for $M = 3$ we have

$$\begin{aligned} \lambda &= \mu - \sum_{k=1}^{N_0} \varepsilon_k \ln(\mu - a_0^k) + \sum_{k=1}^{N_1} \frac{c_1^k}{\mu - a_1^k} + \sum_{k=1}^{N_2} \frac{c_2^k}{\mu - a_2^k} + \frac{1}{2}\sum_{k=1}^{N_2} \frac{(b_2^k)^2}{(\mu - a_2^k)^2} + \\ &+ \sum_{k=1}^{N_3} \frac{d_3^k}{\mu - a_3^k} + \sum_{k=1}^{N_3} \frac{c_3^k b_3^k}{(\mu - a_3^k)^2} + \frac{1}{3}\sum_{k=1}^{N_3} \frac{(b_3^k)^3}{(\mu - a_3^k)^3}, \end{aligned}$$

where N_0, N_1, N_2 and N_3 are arbitrary integers. Corresponding hydrodynamic type system

$$\begin{aligned}
\partial_t a_0^k &= \partial_x[\frac{(a_0^k)^2}{2}+A^0], \qquad \partial_t a_1^k=\partial_x[\frac{(a_1^k)^2}{2}+A^0], \qquad \partial_t b_1^k=\partial_x(b_1^k a_1^k),\\
\partial_t a_2^k &= \partial_x[\frac{(a_2^k)^2}{2}+A^0], \qquad \partial_t b_2^k=\partial_x(b_2^k a_2^k), \qquad \partial_t c_2^k=\partial_x[a_2^k c_2^k+\frac{1}{2}(b_2^k)^2],\\
\partial_t a_3^k &= \partial_x[\frac{(a_3^k)^2}{2}+A^0], \quad \partial_t b_3^k=\partial_x(b_3^k a_3^k), \quad \partial_t c_3^k=\partial_x[a_3^k c_3^k+\frac{1}{2}(b_3^k)^2], \quad \partial_t d_3^k=\partial_x[a_3^k d_3^k+b_3^k c_3^k],
\end{aligned}$$

where $A^0=\sum\limits_{k=1}^{N_0}\varepsilon_k a_0^k+\sum\limits_{k=1}^{N_1}b_1^k+\sum\limits_{k=1}^{N_2}c_2^k+\sum\limits_{k=1}^{N_3}d_3^k$, has following local Hamiltonian structure

$$\begin{aligned}
\partial_t a_0^k &= \frac{1}{2}\partial_x[\frac{1}{\varepsilon_k}\frac{\partial \mathbf{h}_2}{\partial a_0^k}], \qquad \partial_t a_1^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial b_1^k}, \qquad \partial_t b_1^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial a_1^k},\\
\partial_t a_2^k &= \frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial c_1^k}, \qquad \partial_t b_2^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial b_2^k}, \qquad \partial_t c_2^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial a_2^k},\\
\partial_t a_3^k &= \frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial d_3^k}, \qquad \partial_t b_3^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial c_3^k}, \qquad \partial_t c_3^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial b_3^k}, \qquad \partial_t d_3^k=\frac{1}{2}\partial_x\frac{\partial \mathbf{h}_2}{\partial a_3^k},
\end{aligned}$$

where

$$\begin{aligned}
A^n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) &= \frac{1}{n+1}\sum_{k=1}^{N_0}\varepsilon_k(a_0^k)^{n+1}+\sum_{k=1}^{N_1}(a_1^k)^n b_1^k+\sum_{k=1}^{N_2}(a_2^k)^n c_2^k+\frac{n}{2}\sum_{k=1}^{N_2}(a_2^k)^{n-1}(b_2^k)^2\\
&+\sum_{k=1}^{N_3}(a_3^k)^n d_3^k+n\sum_{k=1}^{N_3}(a_3^k)^{n-1}b_3^k c_3^k+\frac{n(n-1)}{6}\sum_{k=1}^{N_3}(a_3^k)^{n-2}(b_3^k)^3.
\end{aligned}$$

The Benney hydrodynamic chain has an infinite series of local Hamiltonian structures (see [35]). The most general Hamiltonian hydrodynamic reductions of the Benney hydrodynamic chain obtained by the Dirac restriction of $(K+1)$st local Hamiltonian structure are determined by the equation of the Riemann surface (42). Its degenerations are given by the more complicated equation

$$\lambda=\frac{\mu^{K+1}}{K+1}+\sum_{k=0}^{K-1}A_{(K)}^k\mu^{K-(k+1)}-\sum_{k=1}^{N_0}\varepsilon_k\ln(\mu-a_0^k)+\sum_{m=1}^{M}\sum_{k=1}^{N_m}\frac{\tilde{a}_m^k}{(\mu-a_m^k)^m}.$$

In this case the number of equations in the corresponding hydrodynamic type system is $N=K+\sum\limits_{m=0}^{M}(m+1)N_m$, where N_m are arbitrary integers. This above equation was found by L.V. Bogdanov and B.G. Konopelchenko (see [4]); if all parameters $\varepsilon_k=0$, this case was considered by I.M. Krichever (see [24]).

4.1 The Zakharov hydrodynamic reduction

The Zakharov hydrodynamic reduction (this is $2N$ component hydrodynamic type system, see [43])

$$u_t^i = \partial_x \left(\frac{(u^i)^2}{2} + \sum \eta^m \right), \qquad \eta_t^i = \partial_x (u^i \eta^i), \quad i = 1, 2, ..., N$$

is associated with the equation of the Riemann surface

$$\lambda = \mu + \sum \frac{\eta^n}{\mu - u^n}.$$

A substitution of the inverse series (at the vicinity of the puncture $\mu^{(k)} = u^k$)

$$\mu^{(k)} = u^k + \eta^k/\lambda + h_k^{(2)}(\mathbf{u}, \boldsymbol{\eta})/\lambda^2 + h_k^{(3)}(\mathbf{u}, \boldsymbol{\eta})/\lambda^3 + ...$$

in the above equation yields *first* N series of conservation law densities, where the *first* N conservation law densities are

$$h_k^{(2)} = \eta^k \left(u^k + \sum_{m \neq k} \frac{\eta^m}{u^k - u^m} \right).$$

A substitution of the above inverse series in (51) yields $2N$ primary commuting flows written in the Riemann invariants (cf. (53))

$$r_{t^k}^i = \frac{1}{\mu^i - u^k} r_x^i, \qquad r_{t^{k+N}}^i = \frac{\eta^k}{(\mu^i - u^k)^2} r_x^i, \qquad i = 1, 2, ..., N. \tag{63}$$

A substitution of the above inverse series in (52) yields a generating function of conservation laws for these $2N$ commuting flows written in the conservative form

$$\mu_{t^k} = \partial_x \ln(\mu - u^k), \qquad \mu_{t^{k+N}} = \partial_x \frac{\eta^k}{u^k - \mu}. \tag{64}$$

A substitution of the above inverse series in (52) yields $2N$ generating functions of commuting flows in the conservative form

$$u_\tau^i = \partial_x \ln(u^i - \mu), \qquad \eta_\tau^i = \partial_x \frac{\eta^i}{u^i - \mu}.$$

Since the Zakharov hydrodynamic reduction has the local Hamiltonian structure

$$u_t^i = \partial_x \frac{\partial \mathbf{h}_2}{\partial \eta^i}, \qquad \eta_t^i = \partial_x \frac{\partial \mathbf{h}_2}{\partial u^i},$$

then the above generating functions of commuting flows are determined by the same Hamiltonian structure, where the Hamiltonian density is given by (cf. (57))

$$\mathbf{h}(\lambda) = \frac{\mu^2(\lambda)}{2} + \sum \eta^k \ln[u^k - \mu(\lambda)].$$

A substitution of the above inverse series in this generating function of conservation law densities yields *second* N series of conservation law densities, where the *second* N conservation law densities are (see [39])

$$h_{k+N}^{(2)}=\frac{(u^k)^2}{2}+\eta^k\ln\eta^k+\sum_{m\neq k}\eta^m\ln(u^k-u^m).$$

Then $2N$ primary commuting flows (63) in the conservative form

$$\begin{aligned}
u_{t^k}^i &= \partial_x\frac{\eta^k}{u^k-u^i}, \qquad \eta_{t^k}^i=\partial_x\frac{\eta^i\eta^k}{(u^k-u^i)^2}, \qquad i\neq k,\\
u_{t^k}^k &= \partial_x\left(u^k+\sum_{m\neq k}\frac{\eta^m}{u^k-u^m}\right), \qquad \eta_{t^k}^k=\partial_x\left[\eta^k\left(1-\sum_{m\neq k}\frac{\eta^m}{(u^k-u^m)^2}\right)\right],\\
u_{t^{k+N}}^i &= \partial_x\ln(u^k-u^i), \qquad \eta_{t^{k+N}}^i=-\partial_x\frac{\eta^i}{u^k-u^i}, \qquad i\neq k,\\
u_{t^{k+N}}^k &= \partial_x\ln\eta^k, \qquad \eta_{t^{k+N}}^k=\partial_x\left(u^k+\sum_{m\neq k}\frac{\eta^m}{u^k-u^m}\right)
\end{aligned}$$

are determined by the Hamiltonian densities

$$u_{t^k}^i=\partial_x\frac{\partial\mathbf{h}_k^{(2)}}{\partial\eta^i}, \quad \eta_{t^k}^i=\partial_x\frac{\partial\mathbf{h}_k^{(2)}}{\partial u^i}, \quad u_{t^{k+N}}^i=\partial_x\frac{\partial\mathbf{h}_{k+N}^{(2)}}{\partial\eta^i}, \quad \eta_{t^k}^i=\partial_x\frac{\partial\mathbf{h}_{k+N}^{(2)}}{\partial u^i}, \quad i,k=1,2,...,N.$$

The first $2N$ primary commuting flows are determined by the extra conservation law (see (18), cf. (59))

$$A_{t^k}^0=\eta_x^k; \tag{65}$$

the second $2N$ primary commuting flows are determined by the extra conservation law (see (18), cf. (59) and the above case)

$$A_{t^{k+N}}^0=u_x^k, \tag{66}$$

where the potential of the Egorov metric $A^0=\Sigma\eta^n$.

Remark: These $2N$ primary commuting flows written in the conservative form also can be obtained by a direct substitution the above inverse series in (64).

Let us write these commuting flows in the potential form

$$d\xi^i=\eta^i\left(1-\sum_{m\neq i}\frac{\eta^m}{(u^i-u^m)^2}\right)dt^i+\left(u^i+\sum_{m\neq i}\frac{\eta^m}{u^i-u^m}\right)dt^{i+N}+\sum_{m\neq i}\left(\frac{\eta^i\eta^m dt^m}{(u^m-u^i)^2}-\frac{\eta^i dt^{m+N}}{u^m-u^i}\right),$$

$$d\xi^{i+N}=\left(u^i+\sum_{m\neq i}\frac{\eta^m}{u^i-u^m}\right)dt^i+\ln\eta^i dt^{i+N}+\sum_{m\neq i}\frac{\eta^m}{u^m-u^i}dt^m+\sum_{m\neq i}\ln(u^m-u^i)dt^{m+N}.$$

In fact we have just two options for derivation of a corresponding solution for the WDVV equation, because the above differentials are symmetric under any the permutability $u^k \leftrightarrow u^n$ and $\eta^k \leftrightarrow \eta^n$.

1. Taking into account (see (**12**), (**65**) and (**66**))

$$\begin{aligned}
\partial_j A^0 &= H_j^2, \quad \partial_j u^i = H_j^{(i)} H_j, \quad \partial_j \eta^i = \tilde{H}_j^{(i)} H_j, \quad \partial_j \frac{\eta^k}{u^k - u^i} = H_j^{(i)} \tilde{H}_j^{(k)}, \\
\partial_j \ln(u^i - u^k) &= H_j^{(i)} H_j^{(k)}, \quad \partial_j \frac{\eta^i \eta^k}{(u^i - u^k)^2} = \tilde{H}_j^{(i)} \tilde{H}_j^{(k)}, \quad \partial_j \left(u^i + \sum_{m \neq i} \frac{\eta^m}{u^i - u^m} \right) = H_j^{(i)} \tilde{H}_j^{(i)}, \\
\partial_j \ln \eta^i &= \left(H_j^{(i)} \right)^2, \qquad \partial_j \left[\eta^i \left(1 - \sum_{m \neq i} \frac{\eta^m}{(u^i - u^m)^2} \right) \right] = \left(\tilde{H}_j^{(i)} \right)^2
\end{aligned}$$

let us choose new flat coordinates

$$a_1 = \eta^1 \left(1 - \sum_{m \neq 1} \frac{\eta^m}{(u^1 - u^m)^2} \right), \quad a_k = \frac{\eta^1 \eta^k}{(u^1 - u^k)^2}, \quad b_1 = u^1 + \sum_{m \neq 1} \frac{\eta^m}{u^1 - u^m}, \quad b_k = \frac{\eta^1}{u^1 - u^k}.$$

Let us replace $t^k \to b_k$ and $t^{k+N} \to a_k$ in the above potential forms taking into account (**30**). Respectively, we must replace $\xi^i \to \partial F / \partial b_i$ and $\xi^{i+N} \to \partial F / \partial a_i$. Then the corresponding solution of the WDVV equation is given by

$$\begin{aligned}
F &= \frac{a_1 (b_1)^2}{2} + \frac{\left(a_1 + \sum\limits_{m \neq 1} a_m \right)}{2} \left[\ln \left(a_1 + \sum_{m \neq 1} a_m \right) - \frac{3}{2} \right] - a_1 \sum_{m \neq 1} a_m \ln b_m \\
&+ b_1 \sum_{m \neq 1} a_m b_m - \frac{1}{2} \sum_{m \neq 1} a_m (b_m)^2 + \sum_{m < k} a_m a_k \ln \frac{a_m - a_k}{b_m b_k} - \sum_{n=1}^{N} (a_k)^2 \ln b_k,
\end{aligned}$$

where $a^k \equiv b_k$ and $b^k \equiv a_k$.

2. Introducing another set of flat coordinates

$$c_1 = \ln \eta^1, \quad c_k = \ln(u^1 - u^k), \quad b_1 = u^1 + \sum_{m \neq 1} \frac{\eta^m}{u^1 - u^m}, \quad b_k = \frac{\eta^k}{u^k - u^1}$$

one can obtain another solution of the WDVV equation connected with the above solution by the Legendre type transformation (see details in [**9**] and [**34**]).

4.2 Three component solutions of the WDVV equation

In this paper without lost of generality we restrict our consideration for simplicity on three component case. It is easy to enumerate all corresponding hydrodynamic reductions of the Benney hydrodynamic chain.

1. The waterbag hydrodynamic reduction associated with the equation of the Riemann surface

$$\lambda = \mu - \sum_{m=1}^{3} \varepsilon_k \ln(\mu - a^k)$$

is considered in the previous section. The corresponding solution (62) of the WDVV equation is given for $N = 3$ also.

2. The first degenerate case

$$\lambda = \mu - \varepsilon \ln(\mu - a) + \frac{b}{\mu - c}$$

can be obtained from the previous waterbag hydrodynamic reduction by a merging of two singular points. The corresponding Egorov hydrodynamic type system

$$a_t = \partial_x \left(\frac{a^2}{2} + b + \varepsilon a \right), \qquad b_t = \partial_x(bc), \qquad c_t = \partial_x \left(\frac{c^2}{2} + b + \varepsilon a \right)$$

possesses the local Hamiltonian structure

$$a_t = \frac{1}{\varepsilon} \partial_x \frac{\partial \mathbf{h}_2}{\partial a}, \qquad b_t = \partial_x \frac{\partial \mathbf{h}_2}{\partial c}, \qquad c_t = \partial_x \frac{\partial \mathbf{h}_2}{\partial b},$$

and the potential of the Egorov metric is $A^0 = b + \varepsilon a$. In this case we seek three commuting flows determined by the conservation law (18) written in the potential form

$$d\xi = A^0 dx + A^1 dt + a dt^1 + b dt^2 + c dt^3.$$

The corresponding canonical Egorov basic set (17) (cf. (59)) in the potential form is

$$d \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon}\left(\frac{b}{a-c} + a\right) & -\frac{b}{a-c} & \ln(a-c) \\ -\frac{b}{a-c} & b + \varepsilon \frac{b}{a-c} & c - \varepsilon \ln(a-c) \\ \ln(a-c) & c - \varepsilon \ln(a-c) & \ln b \end{pmatrix} d \begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix},$$

where the Hamiltonian densities are $\tilde{\mathbf{h}}_1 = a^2/2 + b\ln(a-c)$, $\tilde{\mathbf{h}}_2 = bc - \varepsilon b \ln(a-c)$, $\tilde{\mathbf{h}}_3 = c^2/2 + b(\ln b - 1) + \varepsilon(a-c)[\ln(a-c) - 1]$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. Let, for instance, introduce following flat coordinates

$$a_1 = \frac{1}{\varepsilon}\left(\frac{b}{a-c} + a\right) = \frac{1}{\varepsilon} a^1, \qquad a_2 = -\frac{b}{a-c} = a^3, \qquad a_3 = \ln(a-c) = a^2.$$

Then taking in account (26)

$$\partial_{t^k} a_n = \partial_{t^1} \left(\frac{\partial^2 F}{\partial a^n \partial a^k} \right) \tag{67}$$

the corresponding solution of the WDVV equation can be found (by two quadratures)

$$F = \frac{(a^1)^3}{6\varepsilon} + a^1 a^2 a^3 - a^3 e^{a^2} + \frac{1}{2} a^2 (a^3)^2 - \frac{\varepsilon}{2} a^3 (a^2)^2 + \frac{1}{2}(a^3)^2 \left(\ln a^3 - \frac{3}{2} \right).$$

3. The second degenerate case (the Kodama hydrodynamic reduction (**43**))

$$\lambda = \mu + \frac{c}{\mu - a} + \frac{b^2}{2(\mu - a)^2}$$

can be obtained from the previous waterbag hydrodynamic reduction by a merging of three singular points. The corresponding Egorov hydrodynamic type system

$$a_t = \partial_x \left(\frac{a^2}{2} + c \right), \qquad b_t = \partial_x(ab), \qquad c_t = \partial_x \left(ac + \frac{b^2}{2} \right)$$

possesses the local Hamiltonian structure

$$a_t = \partial_x \frac{\partial \mathbf{h}_2}{\partial c}, \qquad b_t = \partial_x \frac{\partial \mathbf{h}_2}{\partial b}, \qquad c_t = \partial_x \frac{\partial \mathbf{h}_2}{\partial a},$$

and the potential of the Egorov metric is $A^0 = c$. In this case we seek just *two* commuting flows determined by the conservation law (**18**) written in the potential form

$$d\xi = A^0 dx + A^1 dt + a dt^1 + b dt^2,$$

where $t^3 \equiv x$.

The corresponding canonical Egorov basic set (**17**) (cf. (**59**)) in the potential form is

$$d \begin{pmatrix} \xi^3 \\ \xi^2 \\ \xi^1 \end{pmatrix} = \begin{pmatrix} c & b & a \\ b & a - c^2/(2b^2) & c/b \\ a & c/b & \ln b \end{pmatrix} d \begin{pmatrix} t^3 \\ t^2 \\ t^1 \end{pmatrix},$$

where the Hamiltonian densities are $\tilde{\mathbf{h}}_1 = a^2/2 + c \ln b$, $\tilde{\mathbf{h}}_2 = ab + c^2/(2b)$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. Let, for instance, introduce following flat coordinates

$$a_1 = c = a^3, \qquad a_2 = b = a^2, \qquad a_3 = a = a^1.$$

Then (see (**67**)) the corresponding solution of the WDVV equation can be found (by two quadratures)

$$F = \frac{1}{2}(a^1)^2 a^3 + \frac{1}{2} a^1 (a^2)^2 + \frac{1}{2}(a^3)^2 \ln a^2.$$

This solution was found in [9] (see also [2]).

4. This case is the waterbag hydrodynamic reduction associated with the equation of the Riemann surface

$$\lambda = \frac{\mu^2}{2} + A^0 - \sum_{m=2}^{3} \varepsilon_k \ln(\mu - a^k).$$

The corresponding Egorov hydrodynamic type system

$$A_t^0 = \partial_x \left(\sum_{m=2}^{3} \varepsilon_k a^k \right), \qquad a_t^k = \partial_x \left(\frac{(a^k)^2}{2} + A^0 \right), \quad k = 2, 3$$

possesses the local Hamiltonian structure

$$A_t^0 = \partial_x \frac{\partial \mathbf{h}_3}{\partial A^0}, \qquad a_t^k = \frac{1}{\varepsilon_k}\partial_x \frac{\partial \mathbf{h}_3}{\partial a^k}, \quad k = 2, 3,$$

and the potential of the Egorov metric is A^0. In this case we seek just *two* commuting flows determined by the conservation law (**18**) written in the potential form

$$d\xi = A^0 dx + a^2 dt^2 + a^3 dt^3,$$

where $t^1 \equiv x$.

The corresponding canonical Egorov basic set (**17**) (cf. (**59**)) in the potential form is

$$d\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} A^0 & a^2 & a^3 \\ a^2 & \frac{A^0+(a^2)^2/2}{\varepsilon_2} - \frac{\varepsilon_3}{\varepsilon_2}\ln(a^2-a^3) & \ln(a^2-a^3) \\ a^3 & \ln(a^2-a^3) & \frac{A^0+(a^3)^2/2}{\varepsilon_3} - \frac{\varepsilon_2}{\varepsilon_3}\ln(a^2-a^3) \end{pmatrix} d\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix},$$

where the Hamiltonian densities are $\tilde{\mathbf{h}}_3 = (a^2)^3/6 - \varepsilon_3(a^2-a^3)[\ln(a^2-a^3)-1]$ and $\tilde{\mathbf{h}}_3 = (a^3)^3/6 + \varepsilon_2(a^2-a^3)[\ln(a^2-a^3)-1]$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. For instance, the solution of the WDVV equation

$$F = \frac{1}{6}(a^1)^3 + \frac{a^1}{2}[\varepsilon_2(a^2)^2+\varepsilon_3(a^3)^2] + \frac{\varepsilon_2}{24}(a^2)^4 + \frac{\varepsilon_3}{24}(a^3)^4 - \frac{\varepsilon_2\varepsilon_3}{2}(a^2-a^3)^2\left(\ln(a^2-a^3)-\frac{3}{2}\right)$$

can be found (by two quadratures; see (**67**)), where $a^1 = A^0$.

5. This degenerate case (the Kodama hydrodynamic reduction (**43**))

$$\lambda = \frac{\mu^2}{2} + A^0 + \frac{b}{\mu - c}$$

can be obtained from the previous waterbag hydrodynamic reduction by a merging of two singular points. The corresponding Egorov hydrodynamic type system

$$A_t^0 = b_x, \qquad b_t = \partial_x(bc), \qquad c_t = \partial_x\left(\frac{c^2}{2} + A^0\right)$$

possesses the local Hamiltonian structure

$$A_t^0 = \partial_x \frac{\partial \mathbf{h}_3}{\partial A^0}, \qquad b_t = \partial_x \frac{\partial \mathbf{h}_3}{\partial c}, \qquad c_t = \partial_x \frac{\partial \mathbf{h}_3}{\partial b},$$

and the potential of the Egorov metric is A^0. In this case we seek just *one* commuting flow determined by the conservation law (**18**) written in the potential form

$$d\xi = A^0 dx + A^1 dt + c dt^3,$$

where $t^1 \equiv x$, $t^2 \equiv t$, $A^1 \equiv b$.

The corresponding canonical Egorov basic set (17) (cf. (59)) in the potential form is

$$d\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} A^0 & b & c \\ b & bc & A^0 + c^2/2 \\ c & A^0 + c^2/2 & \ln b \end{pmatrix} d\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix},$$

where the Hamiltonian density is $\tilde{\mathbf{h}}_2 = cA^0 + c^3/6 + b(\ln b - 1)$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. Let, for instance, introduce following flat coordinates

$$a_1 = A^0 = a^1, \qquad a_2 = b = a^3, \qquad a_3 = c = a^2.$$

Then (see (67)) the corresponding solution of the WDVV equation can be found (by two quadratures)

$$F = \frac{1}{6}(a^1)^3 + a^1 a^2 a^3 + \frac{1}{6}(a^2)^3 a^3 + \frac{1}{2}(a^3)^2 \left(\ln a^3 - \frac{3}{2}\right).$$

This solution was found in [9] (see also [2] and [5]).

6. This case is the waterbag hydrodynamic reduction associated with the equation of the Riemann surface

$$\lambda = \frac{\mu^3}{3} + A^0\mu + A^1 - \varepsilon \ln(\mu - a).$$

The corresponding Egorov hydrodynamic type system

$$A^0_t = A^1_x, \qquad A^1_t = \partial_x\left(\varepsilon a - \frac{(A^0)^2}{2}\right), \qquad a_t = \partial_x\left(\frac{a^2}{2} + A^0\right)$$

possesses the local Hamiltonian structure

$$A^0_t = \partial_x \frac{\partial \mathbf{h}_4}{\partial A^1}, \qquad A^1_t = \partial_x \frac{\partial \mathbf{h}_4}{\partial A^0}, \qquad a_t = \frac{1}{\varepsilon}\partial_x \frac{\partial \mathbf{h}_4}{\partial a},$$

and the potential of the Egorov metric is A^0. In this case we seek just *one* commuting flow determined by the conservation law (18) written in the potential form

$$d\xi = A^0 dx + A^1 dt + a dt^3,$$

where $t^1 \equiv x$, $t^2 \equiv t$.

The corresponding canonical Egorov basic set (17) (cf. (59)) in the potential form is

$$d\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} A^0 & A^1 & a \\ A^1 & \varepsilon a - (A^0)^2/2 & A^0 + a^2/2 \\ a & A^0 + a^2/2 & (A^1 + aA^0 + a^3/3)/\varepsilon \end{pmatrix} d\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix},$$

where the Hamiltonian density is $\tilde{\mathbf{h}}_2 = aA^1 + a^2A^0/2 + (A^0)^2/2 + a^4/12$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. Let, for instance, introduce following flat coordinates

$$a_1 = A^0 = a^2, \qquad a_2 = A^1 = a^1, \qquad a_3 = a = \frac{1}{\varepsilon}a^3.$$

Then (see (67)) the corresponding solution of the WDVV equation can be found (by two quadratures)

$$F = \frac{1}{2}(a^1)^2 a^2 + \frac{1}{2\varepsilon} a^1 (a^3)^2 + \frac{1}{2}(a^2)^2 a^3 - \frac{1}{24}(a^2)^4 + \frac{1}{6\varepsilon^2} a^2 (a^3)^3 + \frac{1}{60\varepsilon^4}(a^3)^5.$$

7. This case is the so-called Lax reduction (dispersionless limit of the Gelfand–Dikey linear problem; see [9] and [24])

$$\lambda = \frac{\mu^4}{4} + A^0\mu^2 + A^1\mu + A^2 + \frac{3}{2}(A^0)^2.$$

The corresponding Egorov hydrodynamic type system

$$\partial_t h_0 = \partial_x h_1, \qquad \partial_t h_1 = \partial_x \left(h_2 - \frac{(h_0)^2}{2} \right), \qquad \partial_t h_2 = \partial_x(-h_0 h_1)$$

possesses the local Hamiltonian structure

$$\partial_t h_0 = \partial_x \frac{\partial \mathbf{h}_5}{\partial h_2}, \qquad \partial_t h_1 = \partial_x \frac{\partial \mathbf{h}_5}{\partial h_1}, \qquad \partial_t h_2 = \partial_x \frac{\partial \mathbf{h}_5}{\partial h_0},$$

the potential of the Egorov metric is $A^0 = h_0$ and $h_1 = A^1$, $h_2 = A^2 + (A^0)^2$. In this case we seek just *one* commuting flow determined by the conservation law (18) written in the potential form

$$d\xi = h_0 dx + h_1 dt + h_2 dt^3,$$

where $t^1 \equiv x$, $t^2 \equiv t$.

The corresponding canonical Egorov basic set (17) (cf. (59)) in the potential form is

$$d\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_1 & h_2 - (h_0)^2/2 & -h_0 h_1 \\ h_2 & -h_0 h_1 & (h_0)^3/3 - (h_1)^2/2 \end{pmatrix} d\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix},$$

where the Hamiltonian density is $\tilde{\mathbf{h}}_2 = (h_2)^2/2 - h_0(h_1)^2/2 + (h_0)^4/12$. Every three components from *any* column of the above 3×3 matrix can be used as new flat coordinates. Let, for instance, introduce following flat coordinates

$$a_1 = h_0 = a^3, \qquad a_2 = h_1 = a^2, \qquad a_3 = h_2 = a^1.$$

Then (see (67)) the corresponding solution of the WDVV equation can be found (by two quadratures)

$$F = \frac{1}{2}(a^1)^2 a^3 + \frac{1}{2} a^1 (a^2)^2 - \frac{1}{4}(a^2)^2 (a^3)^2 + \frac{1}{60}(a^3)^5.$$

This solution was found in [9].

8. This case associated with the so-called Schwarz-Christoffel map ($\Sigma\varepsilon_k \neq 1$)

$$\lambda = \left(\mu - \sum \varepsilon_k a^k\right) \prod_{m=1}^{3} (\mu - a^k)^{-\varepsilon_k},$$

is considered in the next section.

Remark: In N component case the number of degenerate curves increases. For example, if $N=4$, then we have following list of these curves

$$\begin{aligned}
\lambda &= \mu - \sum_{m=1}^{4}\varepsilon_k \ln(\mu - a^k), \qquad \lambda = \mu - \sum_{m=1}^{2}\varepsilon_k \ln(\mu - a^k) + \frac{b}{\mu - c}, \\
\lambda &= \mu - \varepsilon \ln(\mu - u) + \frac{a}{\mu - c} + \frac{b}{(\mu - c)^2}, \qquad \lambda = \mu + \frac{a_1}{\mu - c_1} + \frac{a_2}{\mu - c_2}, \\
\lambda &= \mu + \frac{a}{\mu - u} + \frac{b}{(\mu - u)^2} + \frac{c}{(\mu - u)^3}, \qquad \lambda = \frac{\mu^2}{2} + A^0 - \sum_{m=1}^{3}\varepsilon_k \ln(\mu - a^k), \\
\lambda &= \frac{\mu^2}{2} + A^0 - \varepsilon \ln(\mu - a) + \frac{b}{\mu - c}, \qquad \lambda = \frac{\mu^2}{2} + A^0 + \frac{a}{\mu - u} + \frac{b}{(\mu - u)^2}, \\
\lambda &= \frac{\mu^3}{3} + A^0\mu + A^1 + \frac{a}{\mu - u}, \qquad \lambda = \frac{\mu^4}{4} + A^0\mu^2 + A^1\mu + A^2 + \frac{3}{2}(A^0)^2 - \varepsilon \ln(\mu - a), \\
\lambda &= \frac{\mu^3}{3} + A^0\mu + A^1 - \sum_{m=1}^{2}\varepsilon_k \ln(\mu - a^k), \qquad \lambda = \left(\mu - \sum \varepsilon_k a^k\right) \prod_{m=1}^{4} (\mu - a^k)^{-\varepsilon_k},
\end{aligned}$$

where $\Sigma\varepsilon_k \neq 1$. In the particular case $\varepsilon_k \equiv \varepsilon$ this is nothing but the Lax hydrodynamic reduction written in a factorized form.

5 Modified Benney hydrodynamic chain

The modified Benney hydrodynamic chain

$$B_t^n = B_x^{n+1} + B^0 B_x^n + nB^n B_x^0, \qquad n = 0,\, 1,\, 2,\, ... \tag{68}$$

is connected with the Benney hydrodynamic chain (1) by an infinite set of the Miura type transformations $A^k(B^0, B^1, ..., B^{k+1})$, which can be derived from comparison of two Riemann mappings (see (34))

$$\lambda = p + B^0 + \frac{B^1}{p} + \frac{B^2}{p^2} + ... = p + B^0 + \frac{A^0}{p + B^0} + \frac{A^1}{(p + B^0)^2} + \frac{A^2}{(p + B^0)^3} + ...,$$

where the generating function of the Miura type transformations is given by $\mu = p + B^0$. Then the Gibbons equation (35) reduces to

$$\lambda_t - (p + B^0)\lambda_x = \frac{\partial \lambda}{\partial p}\left[p_t - \partial_x\left(\frac{p^2}{2} + B^0 p\right)\right]. \tag{69}$$

Thus, hydrodynamic reductions of both hydrodynamic chains coincide. By this reason we consider just two “flat” hydrodynamic reductions.

The modified Benney hydrodynamic chain is a particular case of the Kupershmidt hydrodynamic chain (see (2) and [25]).

Lemma [25]: *The modified Benney hydrodynamic chain*

$$\partial_t B^n_{(\gamma)} = \partial_x B^{n+1}_{(\gamma)} + B^0_{(\gamma)} \partial_x B^n_{(\gamma)} + (n+\gamma) B^n_{(\gamma)} \partial_x B^0_{(\gamma)}, \qquad n = 0,\ 1,\ 2,\ ... \tag{70}$$

is equivalent (68) *under an invertible point transformation.*

Proof: Substitution the series

$$\lambda = p^{1-\gamma} + (1-\gamma) \sum_{k=0}^{\infty} \frac{B^k_{(\gamma)}}{p^{k+\gamma}} \equiv [p + \sum_{k=0}^{\infty} \frac{B^k}{p^k}]^{1-\gamma} \tag{71}$$

in the Gibbons equation (69) yields (70).

Remark: If $\gamma = 1$, then

$$\lambda = \ln p + \sum_{k=0}^{\infty} \frac{B^k_{(1)}}{p^{k+1}} \equiv \ln[p + \sum_{k=0}^{\infty} \frac{B^k}{p^k}].$$

Remark: $B^0_{(\gamma)} \equiv B^0$.

5.1 The first local Hamiltonian structure

If $\gamma = 2$, then the modified Benney hydrodynamic chain

$$\partial_t B^n_{(2)} = \partial_x B^{n+1}_{(2)} + B^0_{(2)} \partial_x B^n_{(2)} + (n+2) B^n_{(2)} \partial_x B^0_{(2)}, \qquad n = 0,\ 1,\ 2,\ ...$$

contains the hydrodynamic reduction

$$c^i_t = \partial_x \left(\frac{(c^i)^2}{2} + B^0 c^i \right), \qquad i = 1, 2, ..., N, \tag{72}$$

associated with the equation of the Riemann surface

$$\lambda = \frac{1}{p} \left(1 + \sum_{i=1}^{N} \varepsilon_i c^i \right) + \sum_{i=1}^{N} \varepsilon_i \ln \left(1 - \frac{c^i}{p} \right), \tag{73}$$

where $B^k_{(2)} = \Sigma \varepsilon_i (c^i)^{k+2}/(k+2)$. This hydrodynamic type system allows the local Hamiltonian structure

$$c^i_t = \frac{1}{2\varepsilon_i} \partial_x \frac{\partial \mathbf{h}_0}{\partial c^i}, \qquad i = 1, 2, ..., N, \tag{74}$$

where the Hamiltonian density $\mathbf{h}_0 = B^1_{(2)} + (B^0_{(2)})^2 \equiv A^0$.

Since A^0 is a potential of the Egorov metric, then the generating function of commuting flows (50) has the conservation law (see (4) and (16))

$$\partial_{\tau(\zeta)} A^0 = \partial_x p(\zeta).$$

Then the generating function of commuting flows (50) (cf. (51))

$$r^i_{\tau(\zeta)} = \frac{p(\zeta)}{p^i(p^i - p(\zeta))} r^i_x, \qquad i = 1, 2, ..., N$$

is connected with the generating function of commuting flows and conservation laws (cf. (52))

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_x \ln \frac{(1 + B^{-1}_{(2)})(p(\lambda) - p(\zeta))}{p(\lambda)p(\zeta)}, \tag{75}$$

where $B^{-1}_{(2)} = \Sigma \varepsilon_i c^i$, and the first conservation law is given by

$$\partial_{\tau(\zeta)} B^0 = \partial_x \ln \frac{p(\zeta)}{1 + B^{-1}_{(2)}}.$$

N commuting flows connected with the corresponding solution of the WDVV equation are determined by the special limit in the above three formulas $p \to c^i$ and $\partial_{\tau(\zeta)} \to \partial_{t^i}$. Then the generating function of commuting flows (cf. (74))

$$c^i_{\tau(\zeta)} = \partial_x \ln \frac{(1 + B^{-1}_{(2)})(c^i - p(\zeta))}{c^i p(\zeta)}$$

is determined by the Hamiltonian density (cf. (57))

$$\tilde{\mathbf{h}} = (1 + B^{-1}_{(2)}) \ln(1 + B^{-1}_{(2)}) - \ln p + \sum \varepsilon_m c^m \ln \frac{c^m - p(\zeta)}{c^m p(\zeta)}. \tag{76}$$

The substitution of the Taylor series (47) in (73) yields an explicit expression for the first conservation law densities $h^{(1)}_i$

$$\varepsilon_i \ln h^{(1)}_i = \sum \varepsilon_m \ln c^i - \sum_{m \neq i} \varepsilon_m \ln(c^i - c^m) - \frac{1 + \sum \varepsilon_m c^m}{c^i}.$$

Taking into account the above expression the substitution $p \to c^i$ in (76) leads to the first auxiliary conservation law densities $\tilde{h}^{(1)}_i$ given by

$$\tilde{h}^{(1)}_i = (1+B^{-1}_{(2)}) \ln(1+B^{-1}_{(2)}) - \sum_{m \neq i} \varepsilon_m (c^i - c^m) \ln(c^i - c^m) + \sum \varepsilon_m [c^i \ln c^i - c^m \ln c^m - c^m \ln c^i] - \ln c^i,$$

which determine N necessary commuting flows (cf. (17), (18) and (59))

$$\begin{aligned} c^i_{t^k} &= \partial_x \ln \frac{(1 + B^{-1}_{(2)})(c^i - c^k)}{c^i c^k}, \\ c^i_{t^i} &= \frac{1}{\varepsilon_i} \partial_x \left[\varepsilon_i \ln(1 + B^{-1}_{(2)}) - \sum_{m \neq i} \varepsilon_m \ln(c^i - c^m) + \left(\sum_{m \neq i} \varepsilon_m - \varepsilon_i \right) \ln c^i - \frac{1 + \sum \varepsilon_m c^m}{c^i} \right]. \end{aligned}$$

A corresponding solution of the WDVV equation can be found in the same way as in the previous examples. In such a case one must choose following new flat coordinates

$$a_1 = \frac{1}{\varepsilon_1}\left[\varepsilon_1\ln(1+B_{(2)}^{-1}) - \sum_{m\neq 1}\varepsilon_m\ln(c^1-c^m) + \left(\sum_{m\neq 1}\varepsilon_m - \varepsilon_1\right)\ln c^1 - \frac{1+\sum\varepsilon_m c^m}{c^1}\right],$$

$$a_k = \ln\frac{(1+B_{(2)}^{-1})(c^1-c^k)}{c^1c^k}.$$

The canonical Egorov basic set is given by the *same* hydrodynamic type systems (**60**), where $c^1(\mathbf{a})$ and $B_{(2)}^{-1}(\mathbf{a})$ are solutions of the algebraic equations

$$B_{(2)}^{-1} = c^1\left[\varepsilon_1 + (1+B_{(2)}^{-1})\sum_{m\neq 1}\frac{\varepsilon_m}{1+B_{(2)}^{-1} - e^{a_m}c^1}\right],$$

$$\sum\varepsilon_m a_m = -\frac{1+B_{(2)}^{-1}}{c^1} + \sum\varepsilon_m\ln\frac{1+B_{(2)}^{-1}}{c^m}.$$

Thus, a corresponding solution of the WDVV equation is given by the *same* expression (see (**62**)).

5.2 The second Hamiltonian structure

If $\gamma = 1$, then the modified Benney hydrodynamic chain

$$\partial_t B_{(1)}^n = \partial_x B_{(1)}^{n+1} + B_{(1)}^0\partial_x B_{(1)}^n + (n+1)B_{(1)}^n\partial_x B_{(1)}^0, \qquad n = 0,\ 1,\ 2,\ ...$$

contains the hydrodynamic reduction (**72**) associated with the equation of the Riemann surface

$$\lambda = p\prod_{k=1}^{N}(p-c^k)^{-\varepsilon_k}, \tag{77}$$

where $B_{(2)}^k = \Sigma\varepsilon_i(c^i)^{k+1}/(k+1)$ and $\Sigma\varepsilon_i = 0$. This hydrodynamic type system (**72**) possesses the local Hamiltonian structure

$$c_t^i = \frac{1}{2}\partial_x\left[\frac{1}{\varepsilon_i}\frac{\partial\mathbf{h}_1}{\partial c^i} - \sum_{n=1}^{N}\frac{\partial\mathbf{h}_1}{\partial c^n}\right], \qquad i = 1, 2, ..., N,$$

where the Hamiltonian density $\mathbf{h}_1 = B_{(1)}^2 + 2B_{(1)}^0B_{(1)}^1 + 2(B_{(1)}^0)^3/3 \equiv A^1$. Above hydrodynamic chain has the second conservation law

$$\partial_t[B_{(1)}^1 + (B_{(1)}^0)^2/2] = \partial_x[B_{(1)}^2 + 2B_{(1)}^0B_{(1)}^1 + 2(B_{(1)}^0)^3/3].$$

Thus, the momentum density $\mathbf{h}_0 = B_{(1)}^1 + (B_{(1)}^0)^2/2 \equiv A^0$ is a potential of the Egorov metric. The generating function of commuting flows and conservation laws (cf. (**75**)) is given by

$$\partial_{\tau(\zeta)}p(\lambda) = \partial_x\ln\frac{\prod(c^i)^{\varepsilon_i}(p(\lambda)-p(\zeta))}{p(\lambda)p(\zeta)},$$

where the first conservation law is given by

$$\partial_{\tau(\zeta)} B^0 = \partial_x \ln \frac{p(\zeta)}{\prod (c^i)^{\varepsilon_i}}.$$

N commuting flows connected with the corresponding solution of the WDVV equation are determined by the special limit in the above three formulas $p \to c^i$ and $\partial_{\tau(\zeta)} \to \partial_{t^i}$. Then the generating function of commuting flows (cf. (74))

$$c^i_{\tau(\zeta)} = \partial_x \ln \frac{\prod (c^n)^{\varepsilon_n} (c^i - p(\zeta))}{c^i p(\zeta)}$$

is determined by the Hamiltonian density (cf. (57))

$$\tilde{\mathbf{h}} = \sum \varepsilon_m c^m \left[\sum \varepsilon_n \ln(c^n - p) + \ln \frac{c^m - p(\zeta)}{c^m p(\zeta)} \right]. \tag{78}$$

The substitution of the Taylor series (47) in (77) yields an explicit expression for the first conservation law densities $h_i^{(1)}$

$$h_i^{(1)} = (c^i)^{1/\varepsilon_i} \prod_{m \neq i} (c^i - c^m)^{-\varepsilon_m/\varepsilon_i}.$$

Taking into account the above expression the substitution $p \to c^i$ in (78) leads to the first auxiliary conservation law densities $\tilde{h}_i^{(1)}$ given by

$$\tilde{h}_i^{(1)} = -\sum_{m \neq i} \varepsilon_m (c^i - c^m) \ln(c^i - c^m) - \sum_{m \neq i} \varepsilon_m c^m \ln c^m + (1 - \varepsilon_i) c^i \ln c^i,$$

which determine N necessary commuting flows (cf. (17), (18) and (59))

$$\begin{aligned} c^i_{t^k} &= \partial_x \ln \frac{\prod (c^m)^{\varepsilon_m} (c^i - c^k)}{c^i c^k}, \\ c^i_{t^i} &= \frac{1}{\varepsilon_i} \partial_x \left[-\sum_{m \neq i} \varepsilon_m \ln(c^i - c^m) + \varepsilon_i \sum_{m \neq i} \varepsilon_m \ln c^m + (\varepsilon_i - 1)^2 \ln c^i \right]. \end{aligned}$$

A corresponding solution of the WDVV equation can be found in the same way as in the previous examples. In such a case one must choose following new flat coordinates

$$\begin{aligned} a_1 &= \frac{1}{\varepsilon_1} \left[-\sum_{m \neq 1} \varepsilon_m \ln(c^1 - c^m) + \varepsilon_1 \sum_{m \neq 1} \varepsilon_m \ln c^m + (\varepsilon_1 - 1)^2 \ln c^1 \right], \\ a_k &= \ln \frac{\prod (c^m)^{\varepsilon_m} (c^1 - c^k)}{c^1 c^k}. \end{aligned}$$

Since the above transformation is invertible

$$c^k = \frac{c^1}{1 + \exp[a_k + \sum \varepsilon_m a_m]}, \qquad \ln c^1 = \sum \varepsilon_m a_m - \sum_{m \neq 1} \varepsilon_m \ln \left(1 + \exp[a_m + \sum \varepsilon_n a_n] \right),$$

then the canonical Egorov basic set is given by (cf. (60))

$$\partial_{t^n} a_1 = \partial_{t^1} a_n, \qquad \partial_{t^n} a_k = \partial_{t^1} \ln(e^{a_k} - e^{a_n}), \qquad k \neq 1, n,$$

$$\partial_{t^n} a_n = \frac{1}{\varepsilon_n} \partial_{t^1} \left[\sum \varepsilon_m a_m - \varepsilon_1 a_n - \ln\left(1 + e^{a_n + \sum \varepsilon_m a_m}\right) - \sum_{m \neq 1, n} \varepsilon_m \ln(e^{a_m} - e^{a_n}) \right].$$

Taking into account $a_k = a^k/\varepsilon_k - \Sigma a^m$ and $a^k = \varepsilon_k(a_k + \Sigma \varepsilon_m a_m)$ the corresponding solution of the WDVV equation (61) can be found in quadratures (the last summation does not contain the index 1)

$$F = \frac{1 - \varepsilon_1}{6\varepsilon_1}(a^1)^3 - \frac{(a^1)^2}{2} \sum_{m \neq 1} a^m + \frac{a^1}{2} \left[\sum_{m \neq 1} \frac{(a^m)^2}{\varepsilon_m} - \left(\sum_{m \neq 1} a^m \right)^2 \right] + P_3(\mathbf{a})$$

$$+ \frac{1}{2} \sum_{m \neq 1} \varepsilon_m \left[\mathrm{Li}_3\left(e^{a^m/\varepsilon_m}\right) + \mathrm{Li}_3\left(e^{-a^m/\varepsilon_m}\right) \right] + \frac{1}{2} \sum_{m < k} \varepsilon_k \varepsilon_m \left[\mathrm{Li}_3\left(e^{a^k/\varepsilon_k - a^m/\varepsilon_m}\right) + \mathrm{Li}_3\left(e^{a^m/\varepsilon_m - a^k/\varepsilon_k}\right) \right],$$

where

$$P_3(\mathbf{a}) = -\frac{1}{6} \left(\sum_{m \neq 1} a^m \right)^3 + \sum_{m \neq 1} \frac{3\varepsilon_m - \varepsilon_1 - 1}{24\varepsilon_m^2} (a^m)^3 + \sum_{n \neq 1} a^n \sum_{m \neq 1} \frac{(a^m)^2}{4\varepsilon_m} - \sum_{m \neq 1} \frac{(a^m)^3}{4\varepsilon_m}.$$

6 Dispersionless limit of the dBKP/Veselov–Novikov hierarchy

Under the invertible transformations

$$B^0_{(\gamma)} = B^0, \quad B^1_{(\gamma)} = B^1 - \frac{\gamma}{2}(B^0)^2, \quad B^2_{(\gamma)} = B^2 - \gamma B^0 B^1 + \frac{\gamma(\gamma+1)}{6}(B^0)^3, \ldots$$

the Kupershmidt hydrodynamic chains (2)

$$\partial_t B^k_{(\gamma)} = \partial_x B^{k+1}_{(\gamma)} + \frac{1}{\beta} B^0 \partial_x B^k_{(\gamma)} + (k + \gamma) B^k_{(\gamma)} B^0_x, \quad k = 0, 1, 2, \ldots$$

are *equivalent* to each other for the *fixed* index β and for an arbitrary index γ. These invertible transformations $B^k_{(\gamma)} = B^k_{(\gamma)}(B^0, B^1, \ldots, B^k)$ can be obtained by a comparison two Riemann mappings (see [32])

$$\lambda = q + \sum_{k=0}^{\infty} \frac{B^k}{q^k} = \left[q^{1-\gamma} + (1 - \gamma) \sum_{k=0}^{\infty} \frac{B^k_{(\gamma)}}{q^{k+\gamma}} \right]^{\frac{1}{1-\gamma}}.$$

If $\gamma = 1$, then the equation of the Riemann mapping (71) reduces to

$$\lambda = \ln q + \sum_{k=0}^{\infty} \frac{B^k_{(1)}}{q^{k+1}} \qquad \Leftrightarrow \qquad \lambda = q \exp\left(\sum_{k=0}^{\infty} \frac{B^k_{(1)}}{q^{k+1}} \right).$$

These above formulas are equivalent up to scaling $\lambda \to \exp \lambda$; since the Gibbons equation (where $q = p^\beta$; see [32])

$$\lambda_t - \left(p^\beta + \frac{B^0}{\beta}\right)\lambda_x = \frac{\partial \lambda}{\partial p}\left[p_t - \partial_x\left(\frac{p^{\beta+1}}{\beta+1} + \frac{B^0}{\beta}p\right)\right]$$

is a *linear* equation with respect to λ, any scaling $\lambda \to \tilde{\lambda}(\lambda)$ is admissible. The Kupershmidt hydrodynamic chains possess infinitely many local Hamiltonian structures and hydrodynamic reductions parameterized (in general case) by the hypergeometric function (see [32]). The Kupershmidt hydrodynamic chains possess two component Egorov hydrodynamic reductions (the ideal gas dynamics, see [29] and [32]). In N component case the Kupershmidt hydrodynamic chains possess the Egorov hydrodynamic reductions if $\beta = 1, 2, \infty$ only.

Egorov (internal) criterion [7]: *The Egorov curvilinear coordinate net is determined by*

$$\beta_{ik}\beta_{kj}\beta_{ji} = \beta_{ij}\beta_{jk}\beta_{ki}, \qquad i \neq j \neq k. \tag{79}$$

Hydrodynamic reductions

$$r_t^i = (q^i + \frac{B^0}{\beta})r_x^i, \qquad i = 1, 2, ..., N$$

of the Kupershmidt hydrodynamic chain are described by the Gibbons–Tsarev system (see [29])

$$\partial_i q^k = q^k \frac{\partial_i B^0}{q^i - q^k}, \qquad \partial_{ik} B^0 = \frac{(q^i + q^k)\partial_i B^0 \partial_k B^0}{(q^i - q^k)^2}, \qquad i \neq k \tag{80}$$

reducible (see [32]) to the canonical form (see [20] and (34), (35), (36), (38))

$$\partial_i \mu^k = \frac{\partial_i A^0}{\mu^i - \mu^k}, \qquad \partial_{ik} A^0 = \frac{\partial_i A^0 \partial_k A^0}{(\mu^i - \mu^k)^2}, \qquad i \neq k \tag{81}$$

by the transformation

$$\mu^i = q^i + B^0, \qquad \partial_i A^0 = q^i \partial_i B^0.$$

It is easy to check that hydrodynamic reductions (38) are the Egorov hydrodynamic reductions. Indeed, taking into account the Combescure transformation (see [42]) $\tilde{H}_i = \mu^i H_i$, where $\tilde{H}_i$ and H_i are different solutions of the linear problem (8), the Gibbons–Tsarev system (81) can be written in the form

$$\beta_{ik} = \frac{H_i^3 H_k^3}{(\tilde{H}_k H_i - H_k \tilde{H}_i)^2} = \frac{\partial_i \tilde{H}_k}{\tilde{H}_i} = \frac{\partial_i H_k}{H_i}, \qquad i \neq k,$$

where $\partial_i A^0 = H_i^2$. It means, that the rotation coefficients β_{ik} are symmetric. However, in general case this is no longer true for an arbitrary index β. Indeed, the Gibbons–Tsarev system (80) can be written via Lame coefficients and rotation coefficients of the curvilinear coordinate nets. Introducing

$$\mu^i + B^0 = \frac{\tilde{H}_i}{H_i}, \qquad \beta_{ik} = \frac{\partial_i \tilde{H}_k}{\tilde{H}_i} = \frac{\partial_i H_k}{H_i}, \qquad i \neq k,$$

one can obtain the Gibbons–Tsarev system written in another form

$$\beta_{ik}=\frac{\psi_i H_i H_k^2[\tilde{H}_i H_k+(\beta-1)H_i\tilde{H}_k-B_0 H_i H_k]}{\beta(\tilde{H}_i H_k-H_i\tilde{H}_k)^2}=\frac{\partial_i\tilde{H}_k}{\tilde{H}_i}=\frac{\partial_i H_k}{H_i}=\frac{\partial_k\psi_i}{\psi_k},\quad i\neq k,$$

where $\partial_i B_0=\psi_i H_i$ and ψ_i is a solution of the adjoint linear problem to (8) (see details in [42]). These rotation coefficients β_{ik} are no symmetric in general case. However, substituting the above expression in (79) one can obtain three exceptional cases $\beta=1$ (the modified Benney hydrodynamic chain), $\beta=2$ (the hydrodynamic chain associated with the dBKP/Veselov–Novikov hierarchy; see [4], see [6]) and $\beta=\infty$ (continuum limit of 2DToda Lattice).

6.1 Waterbag hydrodynamic reduction

In this section we restrict our consideration on the waterbag hydrodynamic reductions for $\beta=2$. The first hydrodynamic chain of the dBKP/Veselov–Novikov hierarchy

$$\partial_t B_{(1)}^k=\partial_x B_{(1)}^{k+1}+\frac{1}{2}B^0\partial_x B_{(1)}^k+(k+1)B_{(1)}^k B_x^0,\quad k=0,1,2,...$$

contains the waterbag hydrodynamic reduction

$$c_t^i=\partial_x\left(\frac{(c^i)^3}{2}+C^0c^i\right),\quad i=1,2,...,N,$$

associated with the equation of the Riemann surface

$$\lambda=2\ln p-\sum_{i=1}^{N}\varepsilon_i\ln\left(1-\frac{(c^i)^2}{p^2}\right),$$

where $B_{(1)}^k=\Sigma\varepsilon_i(c^i)^{2k+2}/(k+1)$ and $C^0=B^0/2$. This hydrodynamic type system allows the local Hamiltonian structure

$$c_t^i=\frac{1}{6\varepsilon_i}\partial_x\frac{\partial\mathbf{h}_0}{\partial c^i},\quad i=1,2,...,N,$$

where the Hamiltonian density $\mathbf{h}_0=B_{(1)}^1+3(B_{(1)}^0)^2/4$. Since this hydrodynamic chain has the couple of conservation laws (4)

$$B_t^0=\partial_x\left(B_{(1)}^1+\frac{3}{4}(B_{(1)}^0)^2\right),\qquad\partial_t\left(B_{(1)}^1+\frac{3}{4}(B_{(1)}^0)^2\right)=\partial_x\left(B_{(1)}^2+2B_{(1)}^0B_{(1)}^1+\frac{3}{4}(B_{(1)}^0)^3\right),$$

then C^0 is a potential of the Egorov metric. Then the generating function of commuting flows (50) has the conservation law (see (4) and (16))

$$\partial_{\tau(\zeta)}C^0=\partial_x p(\zeta).$$

Thus, the generating function of conservation laws and commuting flows is given by (cf. (52) and (75); see, for instance, [4], [6]) and [32])

$$\partial_{\tau(\zeta)}p(\lambda)=\frac{1}{2}\partial_x\ln\frac{p(\lambda)-p(\zeta)}{p(\lambda)+p(\zeta)}.$$

N commuting flows connected with the corresponding solution of the WDVV equation are determined by the special limit in the above three formulas $p \to c^i$ and $\partial_{\tau(\zeta)} \to \partial_{t^i}$. Then the generating function of commuting flows (cf. (74))

$$c^i_{\tau(\zeta)} = \frac{1}{2}\partial_x \ln \frac{c^i - p(\zeta)}{c^i + p(\zeta)}$$

is determined by the Hamiltonian density (cf. (57))

$$\tilde{\mathbf{h}} = \frac{1}{2}\sum \varepsilon_m c^m \ln \frac{c^m - p(\zeta)}{c^m + p(\zeta)}.$$

The substitution of the Taylor series (47) in (77) yields an explicit expression for the first conservation law densities $h_i^{(1)}$

$$h_i^{(1)} = \frac{1}{2}(c^i)^{2(1+\Sigma\varepsilon_m)/\varepsilon_i - 1}\prod_{m\neq i}\left[(c^i)^2 - (c^m)^2\right]^{-\varepsilon_m/\varepsilon_i}.$$

Taking into account the above expression the substitution $p \to c^i$ in (78) leads to the first auxiliary conservation law densities $\tilde{h}_i^{(1)}$ given by

$$\tilde{h}_i^{(1)} = -\frac{1}{2}\sum_{m\neq i}\varepsilon_m(c^i - c^m)\ln(c^i - c^m) - \frac{1}{2}\sum_{m\neq i}\varepsilon_m(c^i + c^m)\ln(c^i + c^m) + \left(1 + \sum_{m\neq i}\varepsilon_m\right)c^i \ln c^i,$$

which determine N necessary commuting flows (cf. (17), (18) and (59))

$$\begin{aligned} c^i_{t^k} &= \frac{1}{2}\partial_x \ln \frac{c^i - c^k}{c^i + c^k}, \\ c^i_{t^i} &= \frac{1}{\varepsilon_i}\partial_x \left[\left(1 + \sum_{m\neq i}\varepsilon_m\right)\ln c^i - \frac{1}{2}\sum_{m\neq i}\varepsilon_m \ln[(c^i)^2 - (c^m)^2]\right]. \end{aligned}$$

A corresponding solution of the WDVV equation can be found in the same way as in the previous examples. In such a case one must choose following new flat coordinates

$$\begin{aligned} a_1 &= \frac{1}{\varepsilon_1}\left[\left(1 + \sum_{m\neq 1}\varepsilon_m\right)\ln c^1 - \frac{1}{2}\sum_{m\neq 1}\varepsilon_m \ln[(c^1)^2 - (c^m)^2]\right], \\ a_k &= \frac{1}{2}\partial_x \ln \frac{c^1 - c^k}{c^1 + c^k}. \end{aligned}$$

Since the above transformation is invertible

$$c^k = -c^1 \tan a_k, \qquad \ln c^1 = \varepsilon_1 a_1 - \sum_{m\neq 1}\varepsilon_m \ln\cosh a_m,$$

then the canonical Egorov basic set is given by (cf. (60))

$$\partial_{t^n} a_1 = \partial_{t^1} a_n, \qquad \partial_{t^n} a_k = \frac{1}{2}\partial_{t^1} \ln \frac{\sinh(a_k - a_n)}{\sinh(a_k + a_n)}, \qquad k \neq 1, n,$$

$$\partial_{t^n} a_n = \frac{1}{\varepsilon_n}\partial_{t^1}\left[\varepsilon_1 a_1 + \left(1 + \sum_{m\neq n}\varepsilon_m\right)\ln\tan a_n + (\varepsilon_1 - \varepsilon_n)\ln\cosh a_n - \frac{1}{2}\sum_{m\neq 1,n}\varepsilon_m A_{mn}\right],$$

where

$$A_{mn} = \ln(\tanh^2 a_m - \tanh^2 a_n) + 2\ln\cosh a_m.$$

The corresponding solution of the WDVV equation (61) can be found in quadratures

$$F = \frac{\varepsilon_1^2}{6}(a_1)^3 + \frac{\varepsilon_1 a_1}{2}\sum_{m\neq 1}\varepsilon_m (a_m)^2$$

$$+\sum_{m\neq 1}\frac{\varepsilon_m(1+\varepsilon_m)}{32}[\mathrm{Li}_3(e^{4a_m}) + \mathrm{Li}_3(e^{-4a_m})] - \frac{2+\sum\varepsilon_n}{8}\sum_{m\neq 1}\varepsilon_m\left[\mathrm{Li}_3\left(e^{2a_m}\right) + \mathrm{Li}_3\left(e^{-2a_m}\right)\right]$$

$$+\frac{1}{16}\sum_{m<k}\varepsilon_k\varepsilon_m[\mathrm{Li}_3\left(e^{2a_k-2a_m}\right) + \mathrm{Li}_3\left(e^{2a_m-2a_k}\right) + \mathrm{Li}_3\left(e^{2a_k+2a_m}\right) + \mathrm{Li}_3\left(e^{-2a_m-2a_k}\right)],$$

where $a^k = \varepsilon_k a_k$ (the last summation does not contain the index 1).

6.2 Modified dBKP/Veselov–Novikov hierarchy

The simplest hydrodynamic chain in the classification of the Egorov integrable hydrodynamic chains (see [33]) is associated with 2+1 nonlinear equation

$$\Omega_{tt} = \Omega_{xy} + \frac{1}{2}\Omega_{xt}^2 - \exp(-2\Omega_{xx})$$

generalizing corresponding 2+1 nonlinear equations from the dKP, dBKP and 2dDTL hierarchies (see details in [33]). However, also this equation is associated with the modified dBKP/Veselov–Novikov hierarchy (details will be published elsewhere). Nevertheless, in this sub-section we derive corresponding solution of the WDVV equation (61) independently. To avoid repetition of the above similar computation, let us briefly (see corresponding details in [33]) describe all necessary actions.

1. The simplest hydrodynamic reductions of the above 2+1 nonlinear equation are given by (see [30] and [31])

$$c_t^i = \partial_x\left[e^{-\mathbf{h}_0}\left(e^{c^i} - e^{-c^i}\right)\right],$$

where the generating function of conservation laws is given by (see [33])

$$p_t = \partial_x\left[e^{-\mathbf{h}_0}\left(e^p - e^{-p}\right)\right].$$

2. The above hydrodynamic type system has the local Hamiltonian structure (see [31])

$$c_t^i = \frac{1}{\varepsilon_i}\partial_x \frac{\partial \mathbf{h}}{\partial c^i}$$

iff $\mathbf{h} \equiv \mathbf{h}_0 = \ln(\Sigma\varepsilon_m \cosh c^m)$.

3. The generating function of commuting flows and conservation laws is given by (see [33])

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_x \ln \frac{e^{p(\lambda)+p(\zeta)} - 1}{e^{p(\lambda)} - e^{p(\zeta)}}. \tag{82}$$

4. The equation of the Riemann surface for the above hydrodynamic reduction is given by (see [30])

$$\lambda = \sum \varepsilon_m \ln \frac{\cosh p - \cosh c^m}{\sinh p}.$$

5. Let us consider conservation law densities at the vicinity of each puncture $p^{(i)} = c^i$ (i.e. $p^{(i)} = c^i + \tilde{\lambda} b^i + ...$, where $\tilde{\lambda}(\lambda)$ is a local parameter). Then applying the Bürmann–Lagrange series (see [27]) one can compute

$$\varepsilon_i \ln b^i = \sum_{m \neq i} \varepsilon_m \ln \frac{\sinh c^i}{\cosh c^i - \cosh c^m}.$$

6. The substitution the above series $p^{(i)} = c^i + \tilde{\lambda} b^i + ...$ in (82) leads to the Hamiltonian hydrodynamic type commuting systems

$$c_{t^i}^i = \partial_x \ln \frac{e^{c^i} - e^{-c^i}}{b^i(\mathbf{c})}, \qquad c_{t^k}^i = \ln \frac{e^{c^i + c^k} - 1}{e^{c^i} - e^{c^k}}, \qquad i \neq k.$$

7. Let us introduce a new set of the flat coordinates

$$a_1 = \ln \frac{e^{c^1} - e^{-c^1}}{b^1(\mathbf{c})}, \qquad a_k = \ln \frac{e^{c^1 + c^k} - 1}{e^{c^1} - e^{c^k}}, \qquad i \neq k.$$

Then the corresponding solution of the WDVV equation is given by

$$F = \frac{\varepsilon_1^2}{6}(a_1)^3 + \frac{\varepsilon_1 a_1}{2}\sum_{m \neq 1} \varepsilon_m (a_m)^2 + \sum_{m \neq 1} \frac{\varepsilon_m(\Sigma\varepsilon_n - 2\varepsilon_m)}{8}[\mathrm{Li}_3(e^{2a_m}) + \mathrm{Li}_3(e^{-2a_m})]$$
$$-\frac{1}{2}\sum_{m<k} \varepsilon_k \varepsilon_m [\mathrm{Li}_3\left(e^{a_k - a_m}\right) + \mathrm{Li}_3\left(e^{a_m - a_k}\right) + \mathrm{Li}_3\left(e^{a_k + a_m}\right) + \mathrm{Li}_3\left(e^{-a_m - a_k}\right)],$$

where $a^k = \varepsilon_k a_k$ (the last summation does not contain the index 1) and the canonical Egorov basic set is given by (cf. with the previous example)

$$\partial_{t^n} a_1 = \partial_{t^1} a_n, \qquad \partial_{t^n} a_k = \partial_{t^1} \ln \frac{\sinh(\frac{a_k + a_n}{2})}{\sinh(\frac{a_k - a_n}{2})}, \qquad k \neq 1, n,$$

$$\partial_{t^n} a_n = \frac{1}{\varepsilon_n}\partial_{t^1}\left[\varepsilon_1 a_1 + \left(\varepsilon_n - \sum_{m \neq n} \varepsilon_m\right) \ln \sinh a_n + \sum_{m \neq 1, n} \varepsilon_m \ln(\cosh a_n - \cosh a_m)\right].$$

7 Conclusion and outlook

In this paper we consider several explicit solution of the WDVV equation expressed via elementary and special functions of flat coordinates. These solutions are connected with the Hamiltonian hydrodynamic reductions of some Egorov hydrodynamic chains such the Benney hydrodynamic chain. Since the Benney hydrodynamic chain has infinitely many Hamiltonian hydrodynamic reductions, then infinitely many corresponding solutions of the WDVV equation can be found. In the same way more complicated solutions of the WDVV equation can be constructed by virtue of the Hamiltonian method of hydrodynamic reductions (see [31]) extracted from more complicated Egorov hydrodynamic chains (see [33]).

The theory of the WDVV equation is closely connected with the theory of N orthogonal curvilinear coordinate nets (see [9], [24], [34], [41], [44]). In this paper we are able completely to ignore computations in Riemann invariants. All computation are made via flat coordinates.

In the sub-section 4.2 the list of three component solutions of the WDVV equation associated with corresponding flat hydrodynamic reductions is presented. However, this list is not complete. Three extra hydrodynamic type systems are connected with the equations of the Riemann surface

$$\lambda = \mu + \frac{b}{\mu - c} - \varepsilon_1 \ln(\mu - c) - \varepsilon_2 \ln(\mu - a), \qquad \lambda = \mu + \frac{c}{\mu - a} + \frac{b^2}{2(\mu - a)^2} - \varepsilon \ln(\mu - a),$$

$$\lambda = \frac{\mu^2}{2} + A^0 + \frac{b}{\mu - c} - \varepsilon \ln(\mu - c).$$

The third case was considered in details in [41]. The first two cases also are embedded in the framework of "logarithmic" deformations of rational curves (see also [41]). If $\varepsilon = 0$, the first case is Example **2**, the second case is Example **3**, the third case is Example **5** from the sub-section 4.2. Thus, ε is a deformation parameter. Moreover, the logarithmic deformation (ε_m are arbitrary deformation parameters) of the Krichever truncation (44)

$$\lambda = \frac{\mu^{K+1}}{K+1} + \sum_{k=0}^{K-1} Q_{K-1-k}(\mathbf{A})\mu^k + \sum_{m=1}^{M}\sum_{n=1}^{N_m} \frac{b_{m,n}}{(\mu - c^m)^n} - \sum_{m=1}^{M} \varepsilon_m \ln(\mu - c^m)$$

preserves just one of two local Hamiltonian structures associated with corresponding hydrodynamic reductions. It means, that if ε_m are small parameters, then the above Riemann surfaces are connected with quasi-Frobenius manifolds, which should be investigated elsewhere.

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